A BOUNDARY CONDITION FOR A HOLOMORPHIC FUNCTION IN A JORDAN REGION TO BE SCHLICHT*

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Let J be a Jordan curve in the complex plane, and denote by G the interior region bounded by J. Then by an arc Λ at a point $\zeta \in J$ we shall mean a simple continuous curve $z = z(t)$ $(0 \leqq t < 1)$ such that $z(t) \in G$ for every t and $\lim_{t \to 1} z(t) = \zeta$.

THEOREM. *Let $f(z)$ be holomorphic in G. Suppose that at every point $\zeta \in J$ there is an arc Λ_ζ along which $f(z)$ tends to a finite limit $\varphi(\zeta)$ as $z \to \zeta$, and that the function $\varphi(\zeta)$ is continuous and schlicht on J. If f does not have the asymptotic value ∞ at any point of J, then f is schlicht in G, and f maps G onto the interior region bounded by the image of J under φ.*[1]

Proof: We show first that f is bounded in G. Assume the contrary. Then, in the notation of cluster-set theory,[2] there exists a point $\zeta_0 \in J$ such that $\infty \in C_G(f,\zeta_0)$. Evidently ∞ does not belong to Noshiro's boundary cluster set[3] $C_J^*(f,\zeta_0)$ formed by means of the arcs Λ_ζ mentioned in our theorem. Hence,

$$\infty \in C_G(f,\zeta_0) - C_J^*(f,\zeta_0),$$

and since f omits the value ∞ in G, it follows from a theorem due to Noshiro[4] that ∞ is an asymptotic value of f at some point of J, contrary to hypothesis.

Denote by K the image of J under φ; K is a Jordan curve. Since f is bounded in G, we know from Lindelöf's theorem[5] that f has no ambiguous point on J, and consequently f has no asymptotic value that does not belong to K.

Denote by H the image of G under f; H is a bounded region. It follows now that f assumes no value lying in the exterior of the Jordan curve K, because otherwise f would have[6] a frontier point of H in the exterior of K as an asymptotic value, which is impossible. Nor can f assume a value lying on K, else f would also assume a value lying in the exterior of K. Therefore H is a subset of the interior of K. But H must actually coincide with the interior of K, because otherwise f would have[6] a frontier point of H in the interior of K as an asymptotic value, which is also impossible.

All that remains to be shown is that f is schlicht in G. Let us assume to the contrary that there is a value $w \in H$ that is assumed by f at two distinct points z_1 and z_2 in G. Consider a rectilinear segment T containing w and lying in H, except for its extremities k_1 and k_2 which lie on K. There exist[7] two Jordan arcs S_1 and S_2 containing z_1 and z_2, respectively, and lying in G, except for their extremities j_1 and j_2 which lie on J, such that f maps each of the open arcs S_1,S_2 topologically onto the open arc T; so that as z tends to j_1 along S_1 or along S_2, f tends to k_1, whereas as z tends to j_2 along S_1 or along S_2, f tends to k_2. (The points j_1 and j_2 are distinct because otherwise f would have an ambiguous point on J. The arcs S_1 and S_2 have the same extremities j_1 and j_2 because $\varphi(\zeta)$ is schlicht on J and f has no ambiguous point on J.) There exists a component G_0 of $G - (S_1 \cup S_2)$ whose frontier is a Jordan curve that is a subset of $S_1 \cup S_2$, and since G is simply connected, we have $G_0 \subset G$. Denote by H_0 the image of G_0 under f. As is readily seen, every frontier point of H_0 lies on T. But since H_0 is a bounded region, this is impossible, and the theorem is proved.

The following example shows that even if f has the asymptotic value ∞ at only one point of J, the theorem may no longer hold.

Let G be the open unit disk and J be the unit circle in the z-plane. Let $Z = g(z)$ map G in a one-to-one conformal manner onto the exterior of the half-strip $\Re(Z) \leqq 0, 0 \leqq \Im(Z) \leqq \pi$ in such a way that the point $z = 1$ corresponds to the point $Z = \infty$. Define $w = f(z)$ to be the function $\exp g(z)$; then $f(z)$ is holomorphic in G. At every point of J except $z = 1$ the function f is continuous, whereas at $z = 1$ both 0 and ∞ are asymptotic values of f. If we define $\varphi(1) = 0$ and

$$\varphi(\zeta) = \lim_{r \to 1^-} f(r\zeta) \quad (\zeta \in J, \zeta \neq 1),$$

then $\varphi(\zeta)$ is continuous and schlicht on J, but f is certainly not schlicht in G.

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[1] Much weaker versions of this result appear in the literature. For example, Bieberbach, L., *Lehrbuch der Funktionentheorie I* (Leipzig: Teubner, 1934), 4th ed. p. 187, assumes that J is rectifiable and that $f(z)$ is holomorphic on $G \cup J$; he refers to the resulting theorem as *der Satz von der Charakteristik des Randes*. Other versions assume at least the continuity of f on $G \cup J$.

[2] See Noshiro, K., *Cluster Sets* (Berlin, 1960).

[3] *Ibid.*, p. 40.

[4] *Ibid.*, p. 43, Theorem 10 (which is evidently valid for G as well as for the unit disk).

[5] See Bieberbach, L., *Lehrbuch der Funktionentheorie II* (Leipzig: Teubner, 1931), p. 21.

[6] See Stoïlow, S., *Oeuvre Mathématique* (Bucureşti, 1964), p. 148.

[7] *Ibid.*, p. 147.