

SOLUTION OF THE HEAWOOD MAP-COLORING PROBLEM*

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One of the most fascinating problems in mathematics is the *four-color conjecture*, and in spite of the fact that we have nothing new to add, a short discussion of the matter is important for our purposes. The fascination of the problem is almost certainly due to the fact that the relevant question may be stated so as to be intelligible to the general public. Are four colors always enough to obtain a *coloring* of the *countries* of *any map* on a sphere? It is only necessary to clarify the italicized words above for the general reader to understand and, if inquisitive, to become interested in the problem.

A *country* must be connected; hence Pakistan, which consists of *two* disjoint parts, does not qualify. The reader begins to realize that we are considering an abstraction which has little resemblance to political reality.

In reference to the term *map* on a sphere, there are no oceans; every point on the sphere is either inside exactly one country or is on the frontiers of two or more countries. Two countries are *adjacent* if they have a common *line* of frontier points. Thus France and Spain are adjacent, but the states of Colorado and Arizona are *not*, in spite of the fact that they have *one* frontier point in common. The negating factor is that there is no common line of frontier points.

A *coloring* of a map on a sphere is an assignment of *one* color to each country so that no pair of adjacent countries is assigned the same color. Thus two countries having the property observed above (Colorado and Arizona) *may* be assigned the same color, but countries like France and Spain *must* be assigned different colors. The minimum number of colors which suffices to color a given map is called the *chromatic number of the map*. The maximum, m , of the chromatic numbers for *all* maps on the sphere is called the *chromatic number of the sphere*. Thus we can be assured that any map on the sphere can be colored by using no more than m colors. The question is: "What is m ?" It is easy to see that there is a map on a sphere that consists of four countries each adjacent to the other three. Hence this map has *four* as its chromatic number. Consequently, $m \geq 4$. This leads to the classical question: "Does $m = 4$?" No one knows the answer. It can be shown, however, that $m \leq 5$.

Many attempts have been made to settle the matter. One of the most notable was made by the English barrister Kempe, who claimed the result in 1880. In 1890 Heawood¹ discovered an error in Kempe's proof and went on to consider the problem for surfaces more complicated than a sphere. The simplest in the hierarchy of such surfaces is a torus, or the surface of a tire. The terms "country," "map," "adjacent," etc., have meaning on such a surface, and Heawood showed that the chromatic number of a torus is *seven*.

The standard topological model of a surface (or orientable two-dimensional manifold) S_p of genus p is a sphere with p handles attached to it. (One may also think of the surfaces of a Swiss cheese with p holes through it.) Thus a torus is

(topologically) a sphere with one attached handle. Heawood's success was due to two facts:

First, he was able to prove the *theorem* that $\chi(S_p)$, the *chromatic number* of S_p , satisfies the inequality

$$\chi(S_p) \leq \left\lceil \frac{7 + \sqrt{1 + 48p}}{2} \right\rceil, \quad (1)$$

if p is *positive*. (The notation $[a]$ stands for the largest integer not greater than a .)

Second, he was able to *exhibit* an example of a map with seven countries on a torus such that each country was adjacent to the other six.

The *theorem* showed that if $p = 1$, then $\chi(S_1) \leq 7$, while the *example* showed that $\chi(S_1) \geq 7$, and thus the matter was settled: $\chi(S_1) = 7$.

Heawood, who wrote and proved in the occasionally casual style of the last century, was under the impression that he had shown equality to hold in (1) for all $p > 0$. A year after his paper appeared, Heffter² drew attention to the incomplete nature of Heawood's arguments and was able to prove that equality holds in (1) for $1 \leq p \leq 6$, and certain additional values of p .

Here the matter stood for about three quarters of a century.

As nearly as can be determined, *within a few years of 1940*, a portion of mathematical folklore was born. This was that equality had been proved in (1) for all $p > 0$. Such a statement is found in Courant and Robbins,³ but communication with the authors has provided no information on the following question: Did their error create the folklore, or vice versa? On the other hand, Feller has told one of us that the folklore, accepted as fact, was known in Göttingen in the early 1930's.

In any event the statement that

$$\chi(S_p) = \left\lceil \frac{7 + \sqrt{1 + 48p}}{2} \right\rceil \quad \text{if } p > 0, \quad (2)$$

came to be known as the *Heawood map-coloring conjecture*.

It is the object of this note to announce that after a lapse of 78 years the Heawood conjecture is settled in the affirmative.

A comment with regard to *method* is worthwhile. First, however, we wish to introduce some *notation*. The term on the right in (1) is used so often that it is convenient to employ the abbreviation

$$H(p) = \left\lceil \frac{7 + \sqrt{1 + 48p}}{2} \right\rceil. \quad (3)$$

The attack is essentially that of Heawood and Heffter, with significant combinatorial refinements. We consider the following problem:

For each $n \geq 3$ determine the smallest integer $\gamma(n)$ for which it is possible to have a map consisting of n countries on the surface $S_{\gamma(n)}$ such that any two countries are adjacent. Note that this implies

$$\chi(S_{\gamma(n)}) \geq n. \quad (4)$$

If we define

$$I(n) = \left\{ \frac{(n-3)(n-4)}{12} \right\} \quad (5)$$

(where the notation $\{a\}$ means the smallest integer not less than a), then it has been shown elsewhere that

$$\gamma(n) \geq I(n) \quad \text{for } n \geq 3. \quad (6)$$

In fact, (6) is called the *complete graph theorem* and

$$\gamma(n) = I(n) \quad \text{for } n \geq 3, \quad (7)$$

the *complete graph conjecture*.

We shall prove that *if (7) is true, then the Heawood map-coloring conjecture is settled in the affirmative.*

Suppose (7) is true. Then for $n \geq 7$ and $I(n) \leq p < I(n+1)$, a direct computation shows that $H(p) = n$. On the other hand, it is easy to see that $\chi(S_p) \geq \chi(S_{I(n)})$, and using (7) and (4), it follows that $\chi(S_p) \geq \chi(S_{I(n)}) = \chi(S_{\gamma(n)}) \geq n$. And now (1) implies that $\chi(S_p) = H(p)$. Since $I(7) = 1$, this implies (2).

The right-hand side of (5) suggests that there may be 12 cases to the problem, depending upon the membership of n in the various residue classes modulo 12. If $n = 12s + k$ with $0 \leq k \leq 11$, we say we are dealing with Case k .

We proceed to provide a short history of the solution, and end with an example from Case 8, one of the last to be solved, and one which illustrates the new techniques required for the last three cases to be settled.

A History of the Solution.—In 1891 Heffter² attacked Case 7; that is, $n = 12s + 7$. Here he was able to show that $\gamma(n) = I(n)$, if $q = 4s + 3$ is prime and the order of 2 in the multiplicative group of integers mod q is either $q - 1$ or $(q - 1)/2$. In addition, as mentioned earlier in dual form, he proved the complete graph conjecture for $n \leq 12$.

In spite of the fact that there is evidence to show that the problem was well known, the first published attack in this century was due to Ringel⁴ in 1952. He proved the equality of $\chi(S_p)$ and the "maximum number of neighboring domains" on S_p , namely, the largest integer ν such that S_p has a map with the property that ν countries in the map are adjacent to each other. The concept was introduced by Heffter.² Moreover, Ringel formally introduced the idea of "orientable scheme," and proved $\gamma(13) = I(13)$.

In 1954 Ringel⁵ solved Case 5. This solution is also found in his book.⁶ It was the first case to be settled completely. In 1961 he succeeded in solving Cases 7, 10, and 3, in that order.⁷

In the spring of 1962 Youngs conducted a seminar on the subject, and his colleague W. Gustin⁸ became interested in what are called the regular cases, namely, those in which $(n-3)(n-4) \equiv 0 \pmod{12}$. He introduced the very powerful weapon of current graphs and announced solutions to Cases 3, 4, and 7, unaware of Ringel's successful solution to the first and last of these cases. Unfortunately, Gustin did not follow his research announcement with details and gave only three examples, one from each of the Cases 3, 4, and 7. It is a pity that

his example in Case 4 is in error. For an exposition of the theory, and a development of the additional basic idea of vortex graphs, the reader may consult Youngs.⁹

In 1963 Terry, Welch, and Youngs found a simpler solution to Case 4, a result that was also obtained independently by Gustin. The proof has not been published. In the same year Terry, Welch, and Youngs¹⁰ solved the remaining regular case, namely Case 0.

In 1963–1964 the theory of vortex graphs was developed by Youngs⁹ and led to a successful solution to Case 1 in collaboration with Gustin.

The next case to fall was Case 9 in 1965, the bulk of the work being done by Gustin. In 1966 Youngs solved Case 6.

This left only Cases 2, 8, and 11.

In early 1967 Ringel found an “index 2” solution (see Youngs⁹) to $n = 12s + 2$, for s odd.

Ringel and Youngs discussed the problem in Berlin in the summer of 1967 and felt that although the rest of Case 2 appeared possible using index 2, they had no idea how to tackle Cases 8 and 11. In a newsletter in 1964 Youngs had already implied some despair with regard to these cases.

We joined forces at the University of California (Santa Cruz) in the fall of 1967, and an attack was launched on $n = 12s + 2$ with s even. However, it was impregnable to an index 2 assault.

In all the cases that are *not* regular, that is, $n \not\equiv 0, 3, 4, 7 \pmod{12}$, there are two parts to the attack. One may be called the “regular” part and the other the “additional-adjacency” problem. This will be made clear in the example below.

The additional-adjacency problem is trivial if $n \equiv 2$ or $5 \pmod{12}$; and though somewhat more difficult if $n \equiv 10 \pmod{12}$, the problem had been solved by Ringel.⁷ The same technique works for $n \equiv 1, 6, \text{ and } 9 \pmod{12}$.

We decided to make the additional-adjacency problem for Case 2 more difficult and to try finding an index 1 solution. It worked. We then began to have hope for Cases 8 and 11. With similar techniques Case 8 fell next and within a few weeks Case 11. It is a pleasure to mention the fact that Richard Guy was in at the finish in December.

Changing the additional-adjacency part of the plan of attack makes the regular part more difficult and, in some cases, impossible for small values of n . In every case that is not regular such a difficulty may arise, and *ad hoc* methods are necessary to settle the matter. At the end of 1967 the only cases in which (7) was in doubt were $n = 18, 20, 23, 30, 35, 47, \text{ and } 59$. Within the past few weeks we have heard from Jean Mayer, Professor of French Literature at the University of Montpellier, and are delighted to learn that he proved (7) for all $n \leq 23$ during 1967. The cases $n = 35, 47, \text{ and } 59$ were solved at the end of February 1968. Finally the case $n = 30$ was solved within the past few days.

An Example.—We propose to illustrate the solution in Case 8 with an example, $n = 32$.

The regular part of the problem is to construct a certain map with 33 countries on a surface S . The countries are identified by the numbers $0, 1, 2, \dots, 29$, and the letters $x, y_0, \text{ and } y_1$. Moreover,

- (1) Each pair of countries identified by numbers is adjacent.
- (2) The country x is adjacent only to every numbered country.
- (3) The country y_0 is adjacent only to the even-numbered countries, y_1 only to the odd-numbered countries.
- (4) At most three countries have a common frontier point.

The Euler formula implies that the genus of S is 67.

The additional-adjacency problem, the reader will see, is to add *one* handle to the surface S and obtain a map in which 32 countries are mutually adjacent. This will show that $\gamma(32) \leq 68$. However, $\gamma(32) \geq I(32) = 68$, by (6). Hence (7) holds for $n = 32$.

We now turn to the regular part and interpret the numbers to be elements of Z_{30} , the additive group of integers mod 30. The reader is referred to Youngs⁹ for definitions of unfamiliar terms.

We use the *current graph* with rotations shown in Figure 1.

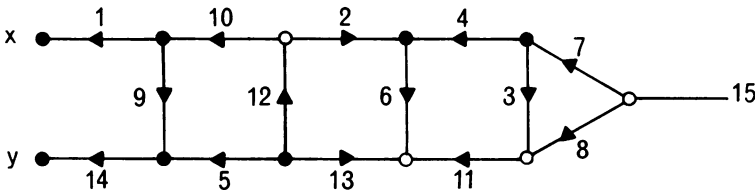


FIG. 1.

Note that:

- (1) There is a singular arc with current 15 and 15 is of order 2 in Z_{30} .
- (2) Kirchoff's current law holds in Z_{30} at each vertex of degree 3.
- (3) The current 1 running into the vortex x generates Z_{30} .
- (4) The current 14 running into the vortex y generates the subgroup of even elements in Z_{30} .
- (5) As shown in Figure 2, there is a single circuit induced by the rotations, and the currents on it contain all the nonzero elements $\pm 1, \dots, \pm 14, 15$ of Z_{30} without repetition.

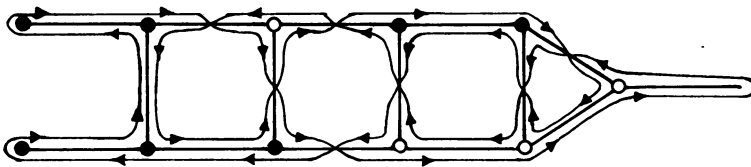


FIG. 2.

This generates an orientable scheme that is the desired map, and hence solves the regular part of the problem (see Youngs⁹).

We stress the fact that there are many solutions to the regular part. In fact, any distribution of currents and rotations that has properties 1-5 above is a solu-

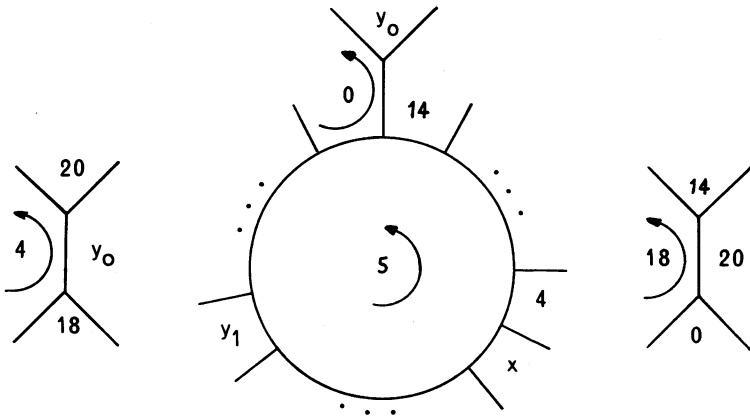


FIG. 3.

tion. However, a solution to the regular part must be chosen in such a way as to make a solution to the additional-adjacency problem possible. This particular solution has the desired property.

To solve the additional-adjacency problem we must look at part of the orientable scheme generated by the vortex graph. The complete permutation in row 0 is obtained by recording the currents associated with the directed arcs on the circuit (writing 29 for -1 , etc.). The pertinent part is contained in the portion

$$0. \quad \dots 14 y_0 16 21 1 x 29 \dots 9 25 12 2 26 \dots$$

Row i is obtained by adding i to each element of Z_{30} in the permutation above, and leaving the position of x and y_0 unchanged; however, y_0 is replaced by y_1 if k is odd. The complete permutations in rows x , y_0 , and y_1 are

$$\begin{array}{l} x. \quad 0 \ 1 \ 2 \ \dots \ 27 \ 28 \ 29 \\ y_0. \quad 0 \ 14 \ 28 \ \dots \ 18 \ 2 \ 16 \\ y_1. \quad 1 \ 15 \ 29 \ \dots \ 19 \ 3 \ 17 \end{array}$$

We need the following portions of rows 4, 5, and 18:

$$\begin{array}{l} 4. \quad \dots 18 y_0 20 \dots \\ 5. \quad \dots y_1 \dots x 4 \dots 14 0 \dots \\ 18. \quad \dots 0 20 14 \dots \end{array}$$

This provides a partial picture (Fig. 3) of the map on S "around" the countries 4, 5, 18, and 0.

We modify the map on S as illustrated in Figure 4. Notice that we have gained the adjacency $(y_0, 5)$ and lost only $(4, y_0)$. Now consider the map of Figure 5 on a torus, in conjunction with Figure 4.

Excise the country 5 from Figure 4 and z from Figure 5. Identify the boundaries of the resulting surfaces in the obvious way. Erase the frontier between y_0 and y_1 , and call the resulting country y . This is the desired map.

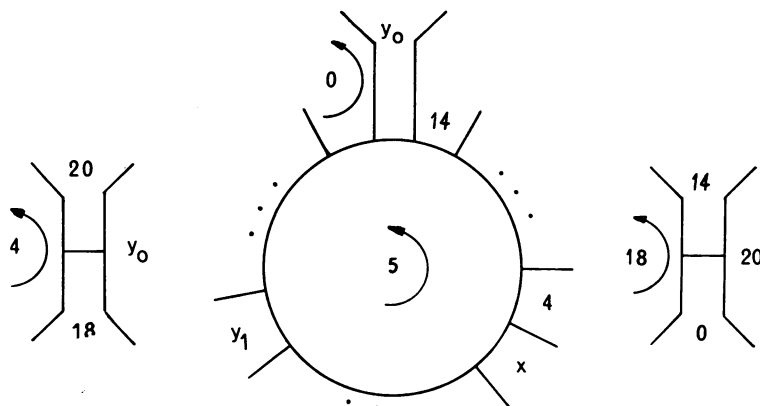


FIG. 4.

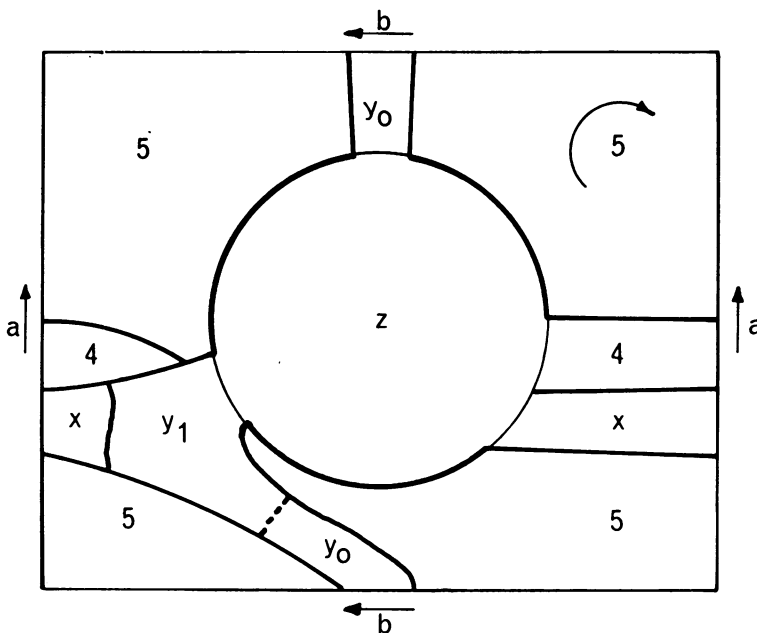


FIG. 5.

The general solution is similar in Cases 2, 8, and 11. The challenge is always to obtain a felicitous “dovetailing” of the regular and additional-adjacency parts of the problem.

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³ Courant, R., and H. Robbins, *What is Mathematics?* (New York: Oxford University Press, 1941), see p. 248.

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⁸ Gustin, W., "Orientable embedding of Cayley graphs," *Bull. Am. Math. Soc.*, **69**, 272–275 (1963).

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¹⁰ Terry, C. M., L. R. Welch, and J. W. T. Youngs, "The genus of K_{12} ," *J. Combinatorial Theory*, **2**, 43–60 (1967).