RANDOM EVOLUTIONS, MARKOV CHAINS, AND
SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

By R. J. Griego and R. Hersh*

UNIVERSITY OF NEW MEXICO, ALBUQUERQUE

Communicated by Einar Hille, December 5, 1968

Abstract.—Several authors have considered Markov processes defined by the motion of a particle on a fixed line with a random velocity $\xi, \eta, \zeta$ or a random diffusivity $\delta, \gamma, \theta$. A "random evolution" is a natural but apparently new generalization of this notion. In this note we hope to show that this concept leads to simple and powerful applications of probabilistic tools to initial-value problems of both parabolic and hyperbolic type. We obtain existence theorems, representation theorems, and asymptotic formulas, both old and new.

Notations.—For $i = 1, \ldots, n$, $T_i(t)$ is a strongly continuous semigroup of bounded linear operators on a fixed Banach space $B$. $A_i$ is the infinitesimal generator of $T_i$.

$v(t)$ is a stationary Markov chain with state space $\{1, \ldots, n\}$ and infinitesimal matrix $Q = (q_{ij})$.

$B$ is the $n$-fold Cartesian product of $B$ with itself. $f$ is a generic element of $B$; $f = (f_i)$, $i = 1, \ldots, n$, a generic element of $B$.

For a sample path $\omega(t)$ of $v$, $\tau_j(\omega)$ is the time of the $j$th jump, and $N(t, \omega)$ is the number of jumps up to time $t$. $\gamma_j(t, \omega)$ is the occupation time in the $j$th state up to time $t$. $E_i$ is the expectation integral on sample paths $\omega$, under the condition $\omega(0) = i$.

Definition 1: A random evolution $M(t, \omega)$ is the product

$$M(t) = T_{\tau(0)}(\tau_1)T_{\tau(1)}(\tau_2 - \tau_1)\ldots T_{\tau(\tau_{N(t)})}(t - \tau_{N(t)}).$$

Definition 2: The "expectation semigroup" $\bar{T}$ is the (matrix) operator on $B$ specified componentwise by $(\bar{T}(t)f)_i = E_i[M(t)f_{\tau(0)}]$, where the integral is taken in the sense of Bochner.

Intuitively, a random evolution is one possible history of a machine capable of $n$ modes of operation, subject to the control of a monkey who flips the control switch at random from one to another of $n$ positions. The expectation semigroup is the average or expected outcome of this procedure.

Before considering any concrete examples, we record three abstract theorems that are easy and basic.

Theorem 1. $\bar{T}(t)$ is a strongly continuous semigroup of linear operators.

Theorem 2. The Cauchy problem for an unknown $n$-vector $\bar{u}(t)$, $t > 0$,

$$\frac{\partial}{\partial t} \bar{u}_t = A_1\bar{u}_t + \Sigma q_{ij}\bar{u}_j, \bar{u}(0) = \bar{f},$$

is solved by

$$\bar{u}(t) = \bar{T}(t)\bar{f}.$$
Remark 1: With appropriate modifications in notation, we may allow $v(t)$ and $T(t)$ to be nonstationary, so that $A_i$ and $q_i$ may depend on $t$ in (1).

Remark 2: Since $B$ is an abstract space, our theorem applies both to the "pure initial-value problem" ($B$ a function space on a domain without boundary) and to the "mixed initial-boundary value problem" ($B$ a function space on a domain with boundary, subject to homogeneous, linear boundary conditions).

**Theorem 3.** If $A_iA_j = A_jA_i$ for $i, j = 1, \ldots, n$,

$$\left(\tilde{T}(t)\tilde{f}\right)_i = E_i\left(\prod_{j=1}^{n} T_j(\gamma_j(t))f_{s(t)}\right).$$

Applications: (i) Parabolic systems: In reference 4, it is shown that if a second-order linear Petrovsky-parabolic system has a fundamental solution matrix with only nonnegative elements, then, in the $i$th equation, only the $i$th unknown is differentiated; the coefficients of the other unknowns are nonnegative. (1) includes the most general system of this type, subject only to the inessential normalization $\Sigma_{i}a_{ij} = 0$ and to the condition that $Q$ is independent of $x$. We can take $B$ to be the function space $C(E)$, $E$ a continuum in some Euclidean space $E_n$, and the $T_{s}(t)$ as the semigroups of $n$ diffusion processes on $E$. We then have a stochastic solution of the most general type of parabolic system that can admit such a solution.

(ii) One-dimensional hyperbolic systems: If $A_i = c_i(\partial/\partial x)$, (1) is a one-dimensional first-order hyperbolic system. (Note that it appears already in diagonal (characteristic) variables as a consequence of its physical derivation.) This case has been studied for constant $c_i$ and $-\infty < x < \infty$. Since in this case $T_{s}(t)f = f(x + c_s t)$, Theorems 2 and 3 give the solution formula

$$\tilde{u}_i(t) = E_i(f_{s(t)}(x + \Sigma_c c_j(\gamma_j(t))))$$

(iii) The Poisson case, and a generalized "telegraph" equation: If $n = 2$, if $A_1 = -A_2 = A$, and if $Q = (\begin{smallmatrix} -a & a \\ -a & a \end{smallmatrix})$, then $u = u_1 + u_2$ solves the second-order Cauchy problem

$$u_{tt} = Au^2 - 2a(t)u_t,$$

$$u(0) = f_1 + f_2, \quad u_t(0) = A(f_1 - f_2).$$

Now $u_1$ and $u_2$ are given by $E_i(T(\gamma_1 - \gamma_2)f_{s(t)})$.

For this $Q$, $N(t)$ is a Poisson process with intensity $a$, and we get as a simple consequence of our general theory the following formula first obtained by Kac for the case $A = \partial/\partial x$.

**Theorem 4.** If $w(t)$ is the unique solution of $w_{tt} = A^2w$ for $t > 0$, $w(0) = f$, $w_t(0) = Ag$, $g$ and $f$ in the domain of $A^2$, then $u(t) = E_t[w(\int_0^t (1)^{N(t)} dt)]$ solves $u_{tt} = A^2u - 2au_t, u(0) = f, u_t(0) = Ag$.

In reference 8, this formula is derived heuristically for $A = \partial/\partial x$, and then verified a posteriori by computing moments. (See also ref. 9 for a simpler and more general verification.)

The a posteriori verification then unexpectedly turns out to be valid for more
general second-order equations \( w_{tt} = Pw \). Our derivation lays bare the probabilistic mechanism that brings about this surprise. It is simply a random alternation between \( T(t) \) and \( T^{-1}(t) = T(-t) \), where the group \( T(t) \) is generated by the square root of \( P \). In this connection, Seeley's work\(^{11} \) on roots of elliptic operators can be used, together with Theorem 23.9.5 of reference 7, to bring under Theorem 4 quite general second-order hyperbolic equations.

(iv) Asymptotic estimates, singular perturbations, and a Gaussian transformation: Finally, we apply Takacs' central limit theorem for occupation times to get an easy new attack on a wide class of singular perturbation problems. If we replace \( a \) by \( a/\epsilon \), \( A \) by \( A/\sqrt{\epsilon} \), then (2) becomes

\[
e_{tt} = A^2u - 2au_t.
\]

Now we find that Takacs' theorem enables us to evaluate the limit of \( u(t) \) as \( \epsilon \downarrow 0 \).

**Theorem 5.** If \( u^* \) satisfies for \( t > 0 \), \( e_{tt}^* = A^2u^* - 2au^*_t \), \( u^*(0) = f, u_t^*(0) = Ag \), \( g \) and \( f \) in the domain of \( A^2 \), then, for all \( t \geq 0 \), \( u^*(t) \) converges weakly to \( u^o(t) = E_{N(0,t/a)}[T(s)f] \) as \( \epsilon \downarrow 0 \). That is, \( u^o \) is obtained, as was \( u^* \), by applying to \( f \) the solution operators \( T(s) \) averaged over a random time \( s \); but in the limit this random time has a normal (Gaussian) distribution, with mean 0 and variance \( t/a \).

**Theorem 6.** The solution of

\[
u_t = \frac{1}{2a} A^2 u, \ (t > 0) \ u(0) = f
\]

is given by

\[
u^o = E_{N(0,t/a)}[T(s)f] = (2\pi t/a)^{-1/2} \int_{-\infty}^{\infty} T(s)f(x) \exp \left(-as^2/2t\right) ds.
\]

This formula is equivalent to Theorem 2 of reference 3, where it appears in a form that masks its probabilistic content. It is a transform formula that shows how to transform a given continuous group \( T(t) \) with generator \( A \) into a holomorphic semigroup with generator \( A^2/2a \).

Theorems 5 and 6 together imply that solutions of (3) converge to solutions of (4).

For \( A^2 = (\partial^2/\partial x^2) \), this result goes back to Hadamard. For \( A^2 \) a self-adjoint second-order elliptic operator, the result was proved by Zlamal.\(^{14} \) A concrete theorem of this type for higher-order equations is proved in reference 10, where it is used to obtain a probabilistic limit theorem. Here, on the contrary, we use a central limit theorem to prove the singular perturbation result, under hypotheses no stronger than the solvability of (3) and existence of a closed, densely defined square root \( A \).

For example, if \( A = i\Sigma(\partial^2/\partial x^2) \), we conclude that the solution of the fourth-order parabolic equation

\[
u_t + \Delta^2 u = 0
\]

is the expected value of random solutions (normally distributed in time) of the
Schrödinger equation \( u_t = i\Delta u \), and that it is the limit of solutions of
\[
\epsilon u_{t\epsilon} + u_{\epsilon} + \Delta^2 u_{\epsilon} = 0
\]  
(vibrating beam equation with damping). The \( u_{\epsilon} \) of (5) is also the expected value of random solutions of Schrödinger's equation, but with a probability distribution defined by the occupation times of a Poisson process with intensity \((1/2\epsilon)\). (Since, in this example, \( A \) is a constant-coefficient partial differential operator, the fact that \( u_{\epsilon} \) converges to \( u \) could also be obtained by Fourier methods; see Theorem 4 of Bobisud.²)

In this work we benefited from the interest, advice, and encouragement of Professor L. H. Koopmans. To Peter D. Lax we owe the useful term, "random evolution." Joe Keller kindly told us of Frisch's work.³ To Andy Schoene we owe thanks for calling our attention to Theorem 23.9.5 in Hille-Phillips.⁷

* This work was supported in part by NSF grants GP-8290 and GP-8856.


