THE DIFFERENTIABILITY OF THE RIEMANN FUNCTION AT CERTAIN RATIONAL MULTIPLES OF π

BY JOSEPH GERVER

COLUMBIA COLLEGE, COLUMBIA UNIVERSITY

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Abstract.—It is shown that a continuous function which Riemann is said to have believed to be nowhere differentiable is in fact differentiable at certain points.

Introduction.—Riemann is reported to have stated,[1, 4] but never proved, that the continuous function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(k^2x)}{k^2}$$

is differentiable nowhere. Kahane[3] renewed the interest in this classical problem in connection with lacunary series, and refers to Weierstrass,[4] who had attempted to prove Riemann's statement, did not succeed, and was then led to his series representing a continuous function nowhere differentiable. To quote from Weierstrass:

Erst Riemann hat, wie ich von einigen seiner Zuhörer erfahren habe, mit Bestimmtheit ausgesprochen (i.J. 1861, oder vielleicht schon früher), dass jene Annahme unzulässig sei, und z.B. bie der durch die unendliche Reihe

$$\sum_{n=1}^{\infty} \frac{\sin(n^2x)}{n^2}$$

dargestellten Function sich nicht bewahrheite. Leider ist der Beweis hierfür von Riemann nicht veröffentlicht worden, und scheint sich auch nicht in seinen Papieren oder mündlich Uberlieferung erhalten zu haben. Dieses ist um so mehr zu bedauern, als ich nicht einmal mit Sicherheit habe erfahren können, wie Riemann seinen Zuhörern gegunüber sich augesadrückt hat.

Riemann's assertion was partially confirmed by Hardy,[2] who proved that the function has no finite derivative at any point $\xi\pi$, where ξ is: (1) irrational; (2) rational of the form $2A/(4B+1)$, where A, B are integers; and (3) rational of the form $(2A+1)/2(B+1)$.

In this paper we shall prove that Riemann's assertion is false by proving the following theorem.

THEOREM 1. *The derivative of*

$$\sum_{k=1}^{\infty} \frac{\sin k^2x}{k^2}$$

exists and is equal to $-1/2$ *at any point* $\xi\pi$, *where* ξ *is a rational number of the* *form* $(2A+1)/(2B+1)$, *i.e., a rational number whose numerator and denominator are odd.*

We shall also extend Hardy's results by proving the following theorem.

THEOREM 2. *The derivative of the Riemann function does not exist at any point* $\xi\pi$, *where* ξ *is a rational number of the form* $(2A+1)/2^N$, *where* N *is an integer* $\geqq 1$.

In order to prove these theorems, we need two lemmas.

LEMMA 1. *Let* μ, ν, *and* λ *be any integers such that*

$$0 < \mu < \nu \leqq \lambda,$$

and let τ *be any real number such that either* $-\pi/2 \leqq \tau \leqq -\pi/\lambda$, *or* $0 \leqq \tau \leqq \pi/2$. *Then*

$$\sum_{k=0}^{\infty}\left(\frac{\sin[(\lambda k+\mu)^2x+\tau]}{(\lambda k+\mu)^2}-\frac{\sin[(\lambda k+\nu)^2x+\tau]}{(\lambda k+\nu)^2}\right)$$

has a right derivative of $\cos\tau(\nu-\mu)/\lambda$ *at* 0.

LEMMA 2. *Let* μ *and* λ *be any integers such that* $0 < \mu \leqq \lambda$, and let τ be any real number such that $0 \leqq \tau < 2\pi$. *Let*

$$f(x) = \sum_{k=0}^{\infty}\frac{\sin[(\lambda k+\mu)^2x+\tau]}{(\lambda k+\mu)^2}.$$

Then, at $x = 0$, *we have:*

$0 \leqq \tau < \pi/2$ *implies that the left derivative of* f *is* $+\infty$,
$\pi/2 < \tau \leqq \pi$ *implies that the right derivative of* f *is* $-\infty$,
$\pi \leqq \tau < 3\pi/2$ *implies that the left derivative of* f *is* $-\infty$,
$3\pi/2 < \tau < 2\pi$ *implies that the right derivative of* f *is* $+\infty$.

In general, the differentiation of the Riemann function at a point P/Q, where P, Q are integers, involves differentiating each subseries formed by taking the summation over those values of k in the same congruence class modulo Q. It is not difficult to prove that for any Q these subseries are all of the form of the functions in Lemmas 1 or 2, if the coordinate system is shifted along the x axis so that $P\pi/Q$ becomes 0. If no more than one of these subseries is of the type in Lemma 2, then it is obviously possible to find the derivative of the entire series by adding the derivatives of all the subseries. If there is more than one subseries of the type in Lemma 2, this is not generally possible, either because one ends up with both unknown right and left derivatives, or because one must add derivatives of $+\infty$ and $-\infty$. In particular, this is true of rational multiples of π of the form $2A/(4B+3)$ and $(2A+1)/2^N(2B+1)$, where $N \geqq 2$. Solutions for these points will have to await a better approximation of the values of the Lemma 2 type series near 0 than are provided by Lemma 2.

Note that Lemmas 1 and 2 simply state in somewhat more generalized form that the Riemann function has a derivative of $-1/2$ at π and $+\infty$ at 0. We will briefly outline the proofs of these simpler results, which in fact formed the basis for the rest of the paper historically.

Let f be the Riemann function, that is,

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin k^2 x}{k^2}.$$

To prove the differentiability at π, fix n. We then show that

$$\left| f(x) - \frac{x-\pi}{2} \right| < \frac{c(x-\pi)}{n},$$

with a suitable constant c, for x sufficiently close to π. Indeed, take

$$|x-\pi| < \frac{\pi}{2n^{14}},$$

and partition the series at those integers k closest to

$$\frac{1}{n}\sqrt{\frac{\pi}{2(x-\pi)}} \quad \text{and} \quad \frac{\pi}{2n(x-\pi)}.$$

Then one shows that the first part approaches 0, the second part approaches $(x-\pi)/2$, and the tail end approaches 0.

On the other hand, to show that the derivative of f at 0 is $+\infty$, we fix n and then show that $f(x) > nx$ for sufficiently small x. For this, let $|x| < \pi/2cn^2$, where c is a suitable constant. Partition the series after $n+1$ at

$$\sqrt{\frac{\pi}{2x}} \quad \text{and} \quad \sqrt{\frac{\pi}{x}}.$$

We then show that the first part is $> nx$, the third is > 0, and the second part is greater than the absolute value of the tail end.

[1] du Bois-Raymond, P., "Versuch einer Classification der willkürlichen Functionen reeler Argumente nach ihren Änderungen in den kleinsten Intervallen," *J. Math.*, **79**, 28 (1875).

[2] Hardy, G. H., "Weierstrass's non-differentiable function," *Trans. Am. Math. Soc.*, **17**, 322–323 (1916).

[3] Kahane, J. P., "Lacunary Taylor and Fourier series," *Bull. Am. Math. Soc.*, **70**, 199–213 (1964).

[4] Weierstrass, K., "Uber continuerliche Funktionen einer reellen Arguments, die fur keinen Werth des letzeren einen bestimmten Differentialquotienten besitzen," *Mathematische Werke*, vol. 2, pp. 71–74.