Fourier Analysis on GL(n,R)

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Communicated by William Feller, November 4, 1969

Abstract. Two problems of Fourier analysis on GL(n,R) are studied. The first concerns the decomposition of the additive Fourier operator in terms of the group representation theory of G. The second concerns the analytic continuation of certain zeta-functions defined on G. It is found that the generalized Gamma functions of Gelfand and Graev arise naturally in the solution of both these problems.

1. Introduction. In this note we state some results concerning the interplay of additive and multiplicative Fourier analysis on the general linear group.

In section 2 we describe the harmonic decomposition of the additive Fourier operator on GL(n,R) with respect to this group's infinite dimensional representation theory. A decomposition of this kind was first obtained by Stein for GL(n,C) for the purpose of constructing new representations of SL(2n,C). Our results for GL(n,R), like Stein's for GL(n,C), show that classical gamma functions arise naturally in this decomposition.

In section 3 we give some results for GL(n,k), k an arbitrary locally compact field, but only with respect to the continuous series of representations. Here the decomposition leads us to gamma functions naturally associated with k. For the other representations of GL(n,k) we offer some conjectures.

In section 4 we consider the analog for n X n real matrix space M(n,R) of the classical M. Riesz potentials. The operator we study is defined on M(n,R) as the (additive) Fourier transform of the operator of multiplication by |det(x)|^{-1}. The problem of realizing this operator as a singular convolution operator is equivalent to the problem of analytically continuing certain distribution-valued functions; a more general formulation of this problem is given by Weil.

It is a pleasure to thank my adviser Professor E. M. Stein for suggesting these problems and giving much helpful advice and encouragement.
In addition to being unitary this modified operator can be shown to be central. Indeed it is the usual Fourier operator modified just so much as to commute with right and left group translations.

Let $\Lambda$ denote the parameterizing space for the irreducible unitary representations appearing in the reduction of the regular representation of $G$. (Henceforth we refer to these representations as representations of the principal series.) For each such representation $T_\lambda^\lambda(y \in G)$ define the Fourier-Mellin transform $T(f)(\lambda)$ of any $f \in L^2(G)$ by

$$T(f)(\lambda) = \int_{G} f(y) T_\lambda(y) \lambda y. $$

Because $\lambda^\lambda$ is central we know a priori that it is given on "the Fourier-Mellin transform side" by a scalar multiplier of absolute value one, i.e., for $f \in L^2(G)$ and almost every $\lambda \in \Lambda$, $T(\lambda^\lambda)(\lambda) = m(\lambda) T(f)(\lambda)$. The problem is to make the factor $m(\lambda)$ completely explicit.

To begin, we state a simple but useful proposition.

**Lemma 1.** Let $G = GL(1, \mathbb{R}) = \mathbb{R}^*$, so that $\Lambda = (\mathbb{R}^2)$ and $T(f)(\lambda)$ is the ordinary Mellin transform. Then $T(\lambda^\lambda)(\lambda) = m(\lambda) T(f)(\lambda)$, where $m(\lambda) = \pi^{\alpha} \{ \Gamma[(1/4) - (is/2)] \Gamma[(1/4) + (is/2)] \}$ if $\lambda(x) = |x|^\alpha$, and $m(\lambda) = i \pi^{\alpha} \{ \Gamma[(3/4) - (is/2)] \Gamma[(3/4) + (is/2)] \}$ if $\lambda(x) = \text{sgn}(x)|x|^\alpha$.

Next consider the case of $G = GL(2, \mathbb{R})$ where already the flavor of the more general situation clearly emerges. Now $\Lambda$ is the union of the sets $\{ (is_1, is_2, e_1, e_2) : s \in \mathbb{R}, e_1 = 0, 1 \}$ and $\{ (n, is, \pm) : n \in \mathbb{Z}^+ \}$. The former set corresponds to characters $\lambda$ of $\mathbb{R}^* \times \mathbb{R}^*$ and parameterizes the continuous series of representations of $G$; the latter set corresponds to characters $\lambda$ of $\mathcal{C}^+$ and parametrize the discrete series. Let $GL_2(2, \mathbb{R})$ denote the subgroup $\{ g \in G : \det(g) > 0 \}$. This is a normal subgroup of index two in $G$ whose representation theory is well known. Thus it is not difficult to describe explicitly the representations $T_\lambda^\lambda$ and the corresponding Plancherel measure $\nu(\lambda) d\lambda$ for $\lambda \in \Lambda$.

**Theorem 1.** $GL(2, \mathbb{R})$. (i) If $\lambda = (i s_1, i s_2, e_1, e_2)$ parameterizes a representation of the continuous series for $G$, then

$$m(\lambda) = \prod_{j=1}^{2} \pi^{e_j} \alpha_j \frac{\Gamma\left[\frac{1 + (2 e_j)/4}{2}\right] - \frac{\lfloor is_j/2\rfloor}{2}}{\Gamma\left[\frac{1 + (2 e_j)/4}{2}\right] + \frac{\lfloor is_j/2\rfloor}{2}}$$

(ii) If $\lambda = (n, is, \pm)$ parameterizes a representation of the discrete series, then

$$m(\lambda) = \pi^{n^+} (2 \pi)^{is} \alpha \frac{\Gamma\left[\frac{n^+ + 1 - is}{2}\right]}{\Gamma\left[\frac{n^+ + 1 + is}{2}\right]}$$

where $n^+ = 2n + 1$ and $n^- = 2n + 2$.

The proof of part (ii) of Theorem 1 is a modification of the proof used by Stein for $GL(n, \mathbb{C})$ where there is no discrete series. This proof requires an explicit knowledge of the Plancherel measure for $G$. The method of proof of part (i) is entirely different; it avoids the Plancherel formula and rests ultimately on Lemma 1.

Before stating the next theorem we recall that gamma functions have been defined for any local field by Gelfand and Graev and studied further by Sally and Taibleson. In particular Sally and Taibleson have shown that $\Gamma(\lambda)$ is a
well-defined function on \( \hat{k}^2 \) which coincides with the local invariant factor \( \rho(\lambda) \) computed by Tate. It is easy to check that the factor \( m(\lambda) \) appearing in part (i) of Theorem 1 is exactly \( \prod_{j=1}^{2} \Gamma(\lambda_j^{1/2} - i) \), where \( \Gamma(\cdot) \) is the gamma function connected with the real field and \( \lambda_j^{1/2} \) is the character \( \lambda_j^{1/2} \) if \( \lambda_j \neq \lambda_j^{1/2} \) and \( \lambda_j^{1/2} \) if \( \lambda_j = \lambda_j^{1/2} \). The factor \( m(\lambda) \) appearing in part (ii) of Theorem 1 is a translate of the gamma function associated with the complex field evaluated at the character of \( C \) parameterizing the representation \( T^\lambda \) of the discrete series.

**THEOREM 2.** \( GL(n, \mathbb{R}) \). Let \( T_{\nu}^{\lambda_m} \), \( m = 0, 1, \ldots, [n/2] \), denote the representation of the principal series for \( G \) induced by the irreducible unitary representation \( \lambda_m \) of the quasi-diagonal subgroup \( H_m \) of \( G \). Then for all \( f \in L^2(G) \) and a.e. \( \lambda_m \in \Lambda \), \( T(\mathfrak{g}^* f)(\lambda_m) = m(\lambda_m) T(f)(\lambda_m) \) with \( m(\lambda_m) \) described as follows. Suppose \( \lambda_m \) is the product of \( m \) discrete representations \( T^{(\pm i, \pm 1, n)} \) of \( GL(2, \mathbb{R}) \) and \( \tau = n - 2m \) characters \( \lambda^{(\tau, \varepsilon_\tau)} \) of \( GL(1, \mathbb{R}) = \mathbb{R}^* \). If \( \Gamma(s, n) = \Gamma(c_n |t|^n) \) and \( \Gamma(s, e_\tau) = \Gamma(|e_\tau|^n) \) are the gamma functions for the fields \( \mathbb{R} \) and \( C \), respectively, then

\[
m(\lambda_m) = \prod_{j=1}^{m} \Gamma(t_j', n_j^{\pm}) \prod_{j=1}^{n} \Gamma(1/2 - is_j, e_\tau)
\]

with \( t_j' = 1/2(-i\tau_j + 1) \), \( n^+ = 2n + 1 \), and \( n^- = 2n + 2 \).

The method of proof of Theorem 2 generalizes the proof of part (i) of Theorem 1 and ultimately uses Lemma 1 and part (ii) of Theorem 1. To carry out the computations involved we chose a special \( f \) whose Fourier transform is sufficiently well-behaved to ensure that \( T(\mathfrak{g}^* f)(\lambda_m) \) is an absolutely convergent integral.

3. **Some Extensions.** For disconnected \( k \) no complete results can be expected until the principal series for \( GL(n, k) \) is better understood. Nevertheless we have the following partial result (the operator \( \mathfrak{g}^* \) can be defined easily enough by making the obvious modifications such as replacing the absolute value everywhere by the natural valuation on \( k \)):

**THEOREM 3.** \( GL(n, k) \), \( (k \) is arbitrary). Let \( T_{\nu}^{\lambda} \) denote the representation of the continuous series for \( G \) induced by the character \( \lambda \) of the diagonal subgroup \( H \). Let \( \Gamma(\lambda_j) \) denote the gamma function for the field \( k \) evaluated at \( \lambda_j e_k^{1/2} \). Then if \( \lambda e_k \) is the direct product of the \( n \)-characters \( \lambda_j e_k^{1/2} \), the factor relating the operators \( T(\mathfrak{g}^* f)(\lambda) \) and \( T(f)(\lambda) \) is given by

\[
m(\lambda) = \prod_{j=1}^{n} \Gamma(\lambda_j^{1/2} - i).
\]

If \( k = C \), Theorem 3 agrees with the result of Stein, and in this case the harmonic decomposition of \( \mathfrak{g}^* \) is completely described by the continuous series. For other representations of \( GL(n, k) \) we hope that a mixture of the above methods can be used to prove the following conjectures.

**Conjecture 1.** Suppose \( n = 2 \) and \( \pi \) is a multiplicative character of a fixed quadratic extension of \( k \). Let \( T_{\nu}^{\pi} \) denote the representation of the discrete series for \( G \) naturally associated with \( \pi \). Then \( m(\pi) = \Gamma(\pi^{1/2} - i) \) with \( \Gamma(\cdot) \) the gamma function defined on this same quadratic extension.

**Conjecture 2.** The analog of Theorem 2 holds—this will follow from Conjecture 1.
4. Riesz Potentials and Zeta-Functions. Let \( M = M(n, \mathbb{R}) \) and let \( C_0^\infty(M) \) and \( s(M) \) denote, respectively, the spaces of smooth compactly supported and smooth rapidly decreasing functions on \( M \). For any \( f \in s(M) \) we define the "zeta-function" \( \zeta(f,s) \) by

\[
\zeta(f,s) = \int_M f(x) |x|^s d^*x
\]

where \( d^*x = dx/|x|^n \). For \( \text{Re}(s) > n - 1 \) this integral is convergent and describes a holomorphic function of the complex variable \( s \). The problem is to effect an analytic continuation of \( \zeta(f,s) \) to the rest of the complex plane. For \( n = 1 \) and \( k \) an arbitrary field, see Tate.9 For \( n > 1 \) and \( k = \mathbb{C} \), Stein used the representation theory of \( GL(n, \mathbb{C}) \) to describe the analytic continuation for \( M(n, \mathbb{C}) \).

For \( M(n, \mathbb{R}) \) the basic result is:

**Theorem 4.** Suppose \( f \in C_0^\infty(M) \). Then the function

\[
\zeta^*(f,s) = \frac{1}{m^*(s)} \zeta(f,s)
\]

initially defined for \( \text{Re}(s) > n - 1 \), has an analytic continuation to all of \( \mathbb{C} \) and for \( \text{Re}(s) < 1 \),

\[
\zeta^*(f,s) = \zeta(f,n - s) = \int_M f(x) |x|^{-s} d^*x.
\]

Here \( m^*(s) = \prod_{i=1}^n \rho(s - i + 1) \) and \( \rho(s) = \pi^{1/2} \Gamma(s/2)/\Gamma(1 - s/2) \).10

The proof of Theorem 4 is based on the functional equation approach used by Tate and a delicate limiting argument used by Stein.11 Note that when \( n = 1 \), \( m^*(s) = \rho(s) \) is Tate's local factor for the real field and this agrees with the factor arising for the classical M. Riesz potentials.12

**Lemma 2.** For \( \text{Re}(s) < 1 \) and \( f \in C_0^\infty(G) \), \( \zeta(f,s) \zeta(g,n - s) = \zeta(f,n - s) \zeta(g,s) \), and thus \( \zeta(f,s) = m^*(s) \zeta(f,n - s) \) for some factor which is independent of the choice of \( f \in C_0^\infty(G) \).

To determine \( m^*(s) \) explicitly we first remark that Lemma 2 is only formally valid for \( f \in s(M) \) since the right and left sides of the functional equation are then defined in nonintersecting half-planes. Thus the following modification.

**Proposition 1.** Let \( \Delta \) denote the differential operator given on \( M(n, \mathbb{R}) \) by \( \det (\partial^2/\partial x_j) \) and set \( \Delta^n = (\Delta)^n \). Then for \( f \in s(M) \)

\[
\zeta(\Delta^{[n/2]} f, s) \zeta(\Delta^{[n/2]} g, n - s) = \zeta(\Delta^{[n/2]} f, n - s) \zeta(\Delta^{[n/2]} g, s)
\]

for all \( s \) with \( n - 1 < \text{Re}(s) < 2[n/2] + 1 \), and thus

\[
\zeta(\Delta^{[n/2]} f, s) = m^*(s) \zeta(\Delta^{[n/2]} f, n - s).
\]

Taking \( f(x) = e^{-\phi(x^2)} \), long computation gives us \( m^*(s) = \prod_{j=1}^n \rho(s - j + 1) \), a function meromorphic in the entire complex plane.

**Proposition 2.** For \( f \in C_0^\infty(G) \), the function \( \zeta^*(f,s) \) has the value \( \zeta(f,n - s) \) when \( \text{Re}(s) < 1 \).
From this proposition follows our Theorem 4.

The method outlined above avoids the representation theory of $GL(n, \mathbb{R})$ and applies equally well to the complex case. In this case $m^*(s) = \prod_{j=1}^{n} \rho(s - 2j + 2)$ where $\rho(s)$ is the gamma function for $\mathbb{C}$ at the character $z \rightarrow |z|^{|z|}$.

3 For $M(\mathbb{C})$ this analog was introduced in ref. 1, pp. 482–490.
5 Cf. ref. 2.
6 Tate, J., *Fourier Analysis in Number Fields and Hecke's Zeta-Functions* (Princeton thesis 1950).
8 Cf. ref. 1, pp. 468–473.
9 Cf. ref. 6.
10 R. Godement has informed me of an independent proof of this theorem; his proof is valid for $k$ an arbitrary locally compact field.
11 Cf. refs. 1 and 6.
12 See, for example, W. Feller, Comm. du séminaire math. de l'université de Lund (1952), dédie à M. Riesz.