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Hilbert Transforms in Yukawan Potential Theory

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ABSTRACT If H denotes the classical Hilbert transform and $Hu(x) = v(x)$, then the functions $u(x)$ and $v(x)$ are the values on the real axis of a pair of conjugate functions, harmonic in the upper half-plane. This note gives a generalization of the above concepts in which the Laplace equation $\Delta u = 0$ is replaced by the Yukawa equation $\Delta u = \mu^2 u$ and in which the Cauchy–Riemann equations have a corresponding generalization. This leads to a generalized Hilbert transform H_μ. The kernel function of this new transform is expressable in terms of the Bessel function K_0. The transform is of convolution type.

Introduction and formulation of two theorems

Boudjelkha and Diaz begin a paper [1] with a caution to prospective readers that the mathematical results to follow stem from a rather naive method. Hadamard characterized this method by the phrase, "He who can do more can do less." The same caution may be appropriate here.

Of concern are solutions $u(x,y)$ of the Yukawa equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \mu^2 u. \tag{1}$$

Here μ is a positive constant. In a previous paper [3], a function $u(x,y)$ satisfying this equation in a region was termed *panharmonic*. Moreover, an ordered pair $[u,v]$ of panharmonic functions was termed a (right) *conjugate pair* if they satisfy the following analog of the Cauchy–Riemann equations

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\mu v, \tag{2a}$$

$$\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} = -\mu u. \tag{2b}$$

Here, I wish to study conjugate pairs that are panharmonic in the upper half-plane. Thus, let $u(x)$ and $v(x)$ be boundary values of such a conjugate pair, $u(x,y)$ and $v(x,y)$. Thus,

$$u(x) = \lim u(x,y) \text{ and } v(x) = \lim v(x,y) \tag{3}$$

as $y \to 0+$. Under suitable restrictions, I show that these boundary values satisfy the following convolution transforms

$$v(x) = \int_{-\infty}^{\infty} h_+(x - x')u(x')dx', \tag{4a}$$

$$u(x) = \int_{-\infty}^{\infty} h_-(x - x')v(x')dx'. \tag{4b}$$

Here the kernel functions h are given by

$$h_\pm(x) = (\mu/\pi)[K_0(\mu x) \pm K_1(\mu x)], \tag{4c}$$

where $K_0(x)$ is the modified Bessel function of the second kind and $K_1(x) = -dK_0(x)/dx$.

In the limit $\mu = 0$ Eqs. (2) become the classical Cauchy–Riemann for conjugate harmonic functions. Moreover, the standard series expansion of the Bessel function K_0 shows that

$$h_\pm(x) \to \pm 1/\pi x \text{ as } \mu \to 0.$$

Hence, relations (**4a**) and (**4b**) become the classical Hilbert transforms.

The first result to be proved is

THEOREM 1. *The generalized Hilbert transform* (4) *is a unitary transformation of the space* $L_2(-\infty, \infty)$. *Moreover an arbitrary transform pair* $[u(x),v(x)]$ *in the space* L_2 *are boundary values (in mean) of a conjugate pair* $[u(x,y),v(x,y)]$ *of panharmonic functions in the open y half-plane* $y > 0$.

In the converse direction the following result is proved.

THEOREM 2. *Let* $[u(x,y),v(x,y)]$ *be a conjugate pair that is panharmonic and uniformly bounded in the closed half-plane* $y \geq 0$. *Then the boundary values* $[u(x),v(x)]$ *are a transform pair of the generalized Hilbert transform.*

Application of Fourier analysis

The Fourier transform formulae are

$$u(x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{ix\bar{x}} w(\bar{x})d\bar{x}, \tag{5a}$$

$$w(\bar{x}) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-ix\bar{x}} u(x)dx. \tag{5b}$$

It is convenient to abbreviate these relations as $u(x) = Tw(\bar{x})$ and $w(\bar{x}) = T^*u(x)$. To insure the simultaneous validity of (**5a**) and (**5b**), attention is confined (at first) to functions of the space $L_2(-\infty, \infty)$. Then the integrals (**5a**) and (**5b**) are known to converge in mean. Moreover, according to Parseval's theorem u and w have the same norm, so

$$\int_{-\infty}^{\infty} |w(x)|^2 dx = \int_{-\infty}^{\infty} |u(x)|^2 dx. \tag{6}$$

A unitary transformation is defined to be a transformation that has an inverse and that preserves the norm. Thus, the Fourier transform T is a unitary transformation.

Let a transformation $v = H_\mu u$ be defined as

$$v(x) = T[(\mu - i\bar{x})\rho^{-1}T^*u(x)], \tag{7a}$$

where $\rho = (\mu^2 + \bar{x}^2)^{1/2}$. The inverse transformation H_μ^{-1} is

$$u(x) = T[(\mu + i\bar{x})\rho^{-1}T^*v(x)]. \tag{7b}$$

To prove this, note that $|(u - i\bar{x})\rho^{-1}| = 1$ so multiplication by $(u - i\bar{x})\rho^{-1}$ does not change the norm. Thus v is expressed as a product of three norm-preserving transformations applied to u, hence v and u have the same norm. It is apparent that (**7a**) substituted in (**7b**) yields the identity transformation. This result proves that H_μ is a unitary transformation. [It will result that H_μ can be represented in the form (**4a**).]

Given an arbitrary function $u(x)$ of L_2, let $v(x)$ be defined by (**7a**). Then define

$$v(x,y) = T[e^{-y\rho}(\mu - i\bar{x})\rho^{-1}T^*u(x)], \quad \textbf{(8a)}$$

$$u(x,y) = T[e^{-y\rho}(\mu + i\bar{x})\rho^{-1}T^*v(x)]. \quad \textbf{(8b)}$$

For $y > 0$ it is apparent that the factor $e^{-y\rho}$ in the integrand decays exponentially at $\pm\infty$. Thereby it is seen that for $y > 0$ the functions $u(x,y)$ and $v(x,y)$ defined by (**8a**) and (**8b**) have partial derivatives of all orders. Moreover (**7b**) gives

$$(\mu + i\bar{x})\rho^{-1}T^*v(x) = T^*u(x) = w(\bar{x})$$

so we may write (**8b**) in the form

$$u(x,y) = T[e^{-y\rho}w(\bar{x})]. \quad \textbf{(8c)}$$

Thus

$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = T[e^{-y\rho}(i\mu\bar{x} + \bar{x}^2 - \rho^2)\rho^{-1}w(\bar{x})] = -\mu v(x,y)$$

$$\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} = T[e^{-y\rho}(-\mu + i\bar{x} - i\bar{x})w(\bar{x})] = -\mu u(x,y).$$

This shows that $u(x,y)$ and $v(x,y)$ defined by (**8**) are a conjugate pair of panharmonic functions in the half-plane $y > 0$.

Now apply the Parseval relation (**6**) to the Fourier transforms (**7**) and (**8**) to obtain

$$\int_{-\infty}^{\infty} |v(x) - v(x,y)|^2 dx = \int_{-\infty}^{\infty} (1 - e^{-y\rho})^2 |w(\bar{x})|^2 d\bar{x},$$

$$\int_{-\infty}^{\infty} |u(x) - u(x,y)|^2 dx = \int_{-\infty}^{\infty} (1 - e^{-y\rho})^2 |w(\bar{x})|^2 dx.$$

Hence, as $y \to 0+$ the function $v(x,y)$ converges in mean to $v(x)$ and $u(x,y)$ converges in mean to $u(x)$. Thus $v(x)$ and $u(x)$ are boundary values. Q.E.D.

Application of the convolution theorem

It is an easy deduction from the Parseval relation (**6**) that if $u_0(x)$ and $u(x)$ both belong to L_2, then

$$T[(T^*u_0)(T^*u)] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} u_0(x - x')u(x')dx'. \quad \textbf{(9)}$$

This is known as the convolution theorem.

To apply the convolution theorem we need the following special Fourier transform (proved in the *Appendix*)

$$T[e^{-y\rho}(\mu \mp i\bar{x})\rho^{-1}] = (2/\pi)^{1/2}\mu[K_0(\mu r) \pm K_1(\mu r)x/r], \quad \textbf{(10)}$$

where $r = (x^2 + y^2)^{1/2}$. Term the right side of (**10**) u_0 and apply the convolution theorem to the relations (**8**). Thus, if $X = x - x'$ and $R = [(x - x')^2 + y^2]^{1/2}$ then

$$v(x,y) = (\mu/\pi) \int_{-\infty}^{\infty} [K_0(\mu R) + K_1(\mu R)X/R]u(x')dx', \quad \textbf{(11a)}$$

$$u(x,y) = (\mu/\pi) \int_{-\infty}^{\infty} [K_0(\mu R) - K_1(\mu R)X/R]v(x')dx'. \quad \textbf{(11b)}$$

Next let $y \to 0+$ in (**11**). Then

$$\lim[K_0(\mu R) \pm K_1(\mu R)x/R] = K_0(\mu X) \pm K_1(\mu X)$$

because $K_0(x)$ is an even function of x and $K_1(x)$ is an odd function of x. From the series expansion of K_0 it is seen that the principal singularity of the limiting kernel is a term $1/\pi X$. It then follows by the same arguments used to analyze the classical Hilbert transform (4) that for almost all x

$$v(x) = (\mu/\pi) \int_{-\infty}^{\infty} [K_0(\mu X) + K_1(\mu X)]u(x')dx', \quad \textbf{(12a)}$$

$$u(x) = (\mu/\pi) \int_{-\infty}^{\infty} [K_0(\mu X) - K_1(\mu X)]v(x')dx'. \quad \textbf{(12b)}$$

The integrals are to be interpreted as Cauchy principal values. This completes the proof of *Theorem 1*.

Application of a Cauchy integral formula

Let $u(x,y)$ and $v(x,y)$ be a conjugate pair of real panharmonic functions and define

$$f = u + iv.$$

I term f a *right regular function*. It is convenient to write $f = f(z)$, where $z = x + iy$. Of course this is not meant to imply that f is a holomorphic function of the complex variable z. Thus suppose $f(z)$ is a right regular function in a compact region bounded by a simple closed contour Γ. Under these conditions, it was shown [3] that the following direct analog of the Cauchy integral formula holds

$$2\pi i f(z) = \int_\Gamma \frac{f(z')}{z' - z} \mu R K_1(\mu R) dz' - \left[\int_\Gamma f(z') K_0(\mu R) dz'\right]^*. \quad \textbf{(13)}$$

Here $z = x + iy$ is a point interior to Γ. $R = |z' - z|$ and $*$ denotes the complex conjugate. But if z is exterior to Γ, then the right side of (**13**) vanishes.

Suppose that $f(z)$ is right regular and uniformly bounded in the closed half-plane $y \geq 0$. Take Γ in (**13**) to be a semicircular contour in this half-plane such that the diameter is along the x-axis. It is well known that $K_0(x)$ and $K_1(x)$ vanish exponentially at infinity. This means that if the center of the contour is fixed and the radius is increased, then the contribution from the circular portion of the contour tends to zero. Thus, if $y > 0$

$$2\pi i f(z) = \int_{-\infty}^{\infty} \frac{f(x')}{x' - z} \mu R K_1(\mu R) dx' - \int_{-\infty}^{\infty} f^*(x') \mu K_0(\mu R) dx', \quad \textbf{(14a)}$$

$$0 = \int_{-\infty}^{\infty} \frac{f(x')}{x' - z^*} \mu R K_1(\mu R) dx' - \int_{-\infty}^{\infty} f^*(x') \mu K_0(\mu R) dx'. \quad \textbf{(14b)}$$

Adding these last two equations and separating into real and imaginary parts, one again obtains the formula (**11a**) for

$v(x,y)$ and the formula (**11b**) for $u(x,y)$. Then, allowing y to approach zero, one obtains (**12a**) and (**12b**). But now these relations hold for all x rather than for almost all x because $u(x)$ and $v(x)$ are smooth functions. This completes the proof of *Theorem 2*.

Conjugate harmonic functions in three-space

In another paper [2], I studied a different but greater generalization of the Cauchy–Riemann equations. This is the system (in ref 2 y and z are interchanged)

$$\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = -i\frac{\partial V}{\partial z}, \qquad (15a)$$

$$\frac{\partial V}{\partial y} - \frac{\partial U}{\partial x} = -i\frac{\partial U}{\partial z} \qquad (15b)$$

for a pair of functions $[U(x,y,z),V(x,y,z)]$ of three real variables (x,y,z). If these functions have second derivatives it is clear that they are harmonic. Then $[U,V]$ is termed a *conjugate pair of harmonic functions*. In general such functions are complex valued.

In the cited paper the following integral formulae were found to relate conjugate harmonic functions in the half-space $y \geq 0$ to their boundary values on the (x,z) plane.

$$V(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' \times \int_{-\infty}^{\infty} \frac{[(x-x') - i(z-z')]U(x',0,z')}{[(x-x')^2 + y^2 + (z-z')^2]^{3/2}}\, dz', \qquad (16a)$$

$$U(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' \times \int_{-\infty}^{\infty} \frac{[-(x-x') - i(z-z')]V(x',0,z')}{[(x-x')^2 + y^2 + (z-z')^2]^{3/2}}\, dz'. \qquad (16b)$$

Setting $y = 0$ gives formulae that are a two-dimensional generalization of the Hilbert transforms.

Now let $[u(x,y),v(x,y)]$ be a pair of conjugate panharmonic functions. Define

$$U(x,y,z) = e^{-i\mu z}u(x,y), \quad V(x,y,z) = e^{-i\mu z}v(x,y). \qquad (17)$$

Then it is clear that $[U,V]$ so defined is a pair of conjugate harmonic functions. Thus, substitute these functions into the integral formulae (**14**) and let $z = 0$ to obtain

$$v(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{[(x-x') + iz']e^{-i\mu z'}dz'}{[(x-x')^2 + y^2 + (z')^2]^{3/2}} \times u(x,0)dx', \qquad (18a)$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{[-(x-x') + iz']e^{-i\mu z'}dz'}{[(x-x')^2 + y^2 + (z')^2]^{3/2}} \times v(x',0)dx'. \qquad (18b)$$

The integral relations (**18**) are the same as the relations (**11**). To prove this, one uses formula (**20**) in the *Appendix* to express the kernel functions of (**18**) in terms of the Bessel function K_0. This is seen to give

$$\int_{-\infty}^{\infty} \frac{(iz' \pm x)e^{-i\mu z'}dz'}{[r^2 + (z')^2]^{3/2}} = 2\mu[K_0(\mu r) \pm K_1(\mu r)x/r]. \qquad (18c)$$

Appendix—Evaluation of a Fourier integral

Using polar coordinates, one finds that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{i(x\bar{x} + z\bar{z})}e^{-y(\bar{x}^2 + \bar{z}^2)^{1/2}}d\bar{x}d\bar{z}}{2\pi(\bar{x}^2 + \bar{z}^2)^{1/2}} = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}.$$

Operating on both sides with $\int_{-\infty}^{\infty} e^{-i\mu z} \; -dz$ and using the Fourier integral theorem gives

$$\int_{-\infty}^{\infty} \frac{e^{ix\bar{x}}e^{-y(\bar{x}^2 + \mu^2)^{1/2}}d\bar{x}}{(\bar{x}^2 + \mu^2)^{1/2}} = \int_{-\infty}^{\infty} \frac{e^{-i\mu z}dz}{(x^2 + y^2 + z^2)^{1/2}}. \qquad (19)$$

But the modified Bessel function K_0 is given by the well-known formula

$$K_0(\mu r) = \int_0^{\infty} \frac{\cos \mu z}{(r^2 + z^2)^{1/2}}\, dz. \qquad (20)$$

Thus (**19**) and (**20**) give the Fourier integral

$$T(e^{-y\rho}/\rho) = (2/\pi)^{1/2}K_0(\mu r) \qquad (21)$$

where $\rho = (\bar{x}^2 + \mu^2)^{1/2}$ and $r = (x^2 + y^2)^{1/2}$. Differentiating (**21**) with respect to x gives

$$T(-ixe^{-y\rho}/\rho) = (2/\pi)\mu K_1(\mu r)x/r \qquad (22)$$

where K_1 is the modified Bessel function of first order. Then adding (**21**) and (**22**) yields the Fourier integral (**10**). Q.E.D.

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