In general the results of the digestion experiments here reported are in accord with conclusions drawn from earlier studies of the digestibility of wheat flours. The digestibility of the 70% (95% patent) flour was the highest, that of the 54% flour was slightly greater than that of the 85% ("whole wheat") flour, while the digestibility of the 100% (graham) flour was lowest of all those studied. Since the flavor of bread made with the different flours varies, the use of different kinds for bread making is an easy way of giving variety to the diet.

[1] U. S. Dept. Agr., Bur. Crop Estimates, *Monthly Crop Rept.*, **3**, 1917, No. 10 (99).

[2] U. S. Dept. Agr., *Office Expt. Sta. Bull.* **85**, 1900 (32–33); *Bull.* **101**, 1901 (33); *Bull.* **126**, 1903 (29, 45); *Bull.* **143**, 1904 (32); *Bull.* **156**, 1905 (36).

[3] U. S. Dept. Agr., *Bull.* **310**, 1915; **617**, 1919; **717**, 1919.

[4] U. S. Dept. Agr., *Bull.* **470**, 1916 (7); **525**, 1917 (4).

[5] *Connecticut Storrs Sta. Rpt.*, 1899 (104).

[6] U. S. Dept. Agr., *Bull.* **310**, 1915.

THE MATHEMATICAL THEORY OF THE APPORTIONMENT OF REPRESENTATIVES[1]

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The Problem.—The exact quota to which each state is theoretically entitled on the basis of population usually involves a fraction. *The problem is, to replace these exact quotas by whole numbers* in such a way that the resulting injustice (due to adjustment of the fractions) shall be as small as possible.

This problem has been the subject of violent debate in Congress for the past one hundred years, a new method of apportionment having been proposed after almost every decennial census. None of these methods, however, possesses any satisfactory mathematical justification. The need of a strictly mathematical treatment of the problem having been called to the writer's attention by Dr. J. A. Hill, Chief Statistician of the Bureau of the Census, the following solution has been worked out on the basis of two very simple postulates. The new method may be called the *Method of Equal Proportions*.

Let N be the total number of representatives, A, B, C, ... the populations of the several states, and a, b, c, ... the number of representatives assigned to each.

Fundamental Principle.—In a satisfactory apportionment between two states (A greater than B), we shall agree that A/a and B/b should be as nearly equal as possible; also a/A and b/B; also A/B and a/b; also B/A and b/a.

Now to say that two quantities are "nearly equal" may be interpreted to mean: *either*, that the difference between the quantities is nearly zero; *or*, that the ratio between them is nearly one.

(Here the difference "between" two quantities means the larger minus the smaller. Similarly, the ratio "between" two quantities means the larger divided by the smaller.)

If we adopt the "difference" interpretation, we have:

POSTULATE I*a*. *The difference between* A/a *and* B/b; or

POSTULATE I*b*. *The difference between* a/A *and* b/B; or

POSTULATE I*c*. *The difference between* A/B *and* a/b; or

POSTULATE I*d*. *The difference between* B/A *and* b/a; *should be as near zero as possible.*

If, on the other hand, we adopt the "ratio" interpretation, we have:

POSTULATE I. *The ratio between* A/a *and* B/b (*or the ratio between* a/A *and* b/B; *or the ratio between* A/B *and* a/b; *or the ratio between* B/A *and* b/a; *all of which have the same value*) *should be as near unity as possible.*

Since there is no way of choosing, mathematically, between Postulates I*a* and I*b* or between Postulates I*c* and I*d*, and since these four demands lead to four different results, we shall reject all four of them and *adopt Postulate I as the proper interpretation of our Fundamental Principle.*

The case of two states is thus disposed of.

For the case of three or more states, one further principle is required, which we state as follows:

POSTULATE II. *In a satisfactory apportionment, there should be no pair of states which is capable of being "improved" by a transfer of representatives within that pair*—the word "improvement" being understood in the sense implied by the test already adopted for the case of two states, and the rare cases of "no choice" being decided in favor of the larger state.

From these two postulates the following theorem can then be deduced:

THEOREM I. *For any given values of* A, B, C, ... *and* N, *there will always be one and only one satisfactory apportionment in the sense defined by Postulates I and II.* No further principles are required.

A working rule for computing this "best" apportionment in any given case is found to be as follows:

Working Rule.—Multiply the population of each state by as many of the numbers

$$\text{Inf.},\ 1/\sqrt{1 \times 2},\ 1/\sqrt{2 \times 3},\ 1/\sqrt{3 \times 4},\ \ldots$$

as may be necessary, and record each result, together with the name of the state, on a small card. Arrange these cards according to the magnitude of the numbers recorded upon them, from the largest to the smallest, thus forming a *priority list* for the given states (the cards marked "Inf." being placed at the head of the list, arranged among themselves in order of magnitude of the populations of the states). Finally, *assign the representatives, from the 1st to the Nth, to the several states in the order in which the names of the states occur in this priority list.* (It should be noted that this method satisfies automatically the constitutional requirement that every state shall have at least one representative.)

This method may be called the "*method of the geometric mean,*" since the "multipliers" are the reciprocals of the geometric means of consecutive integers.

The solution of the problem is thus complete.

Alternative Methods.—If we had adopted Postulate I*a* or I*b* we should have been led, in like manner, to two other methods which may be called the *method of the harmonic mean* (I*a*), and the *method of the arithmetic mean* (I*b*), since the "multipliers" in the working rules are as follows:

$$\text{(I}a\text{)} \quad \text{Inf.,} \quad \frac{1+2}{2(1\times 2)}, \quad \frac{2+3}{2(2\times 3)}, \quad \frac{3+4}{2(3\times 4)}, \quad \ldots$$

$$\text{(I}b\text{)} \quad 2, \quad 2/3, \quad 2/5, \quad 2/7, \quad \ldots$$

It can be shown that method I*a* favors the smaller states more than method I does, while method I*b* favors the larger states more than method I does. Since there is no mathematical reason for adopting either of the two Postulates I*a* and I*b* to the exclusion of the other, both should be rejected.

Postulates I*c* and I*d* also determine two distinct methods, which may be called *the two methods of similarity ratios.* It can be shown that I*c* favors the small states even more than I*a* does, while I*d* favors the large states even more than I*b* does, so that both should be rejected.

Each of these four methods violates three of the four conditions expressed in our Fundamental Principle, while the method of the geometric mean satisfies all these conditions simultaneously.

The following further methods are suggested by the Theory of Least Squares.

In a theoretically perfect apportionment, A/a would be equal to P/N, and a/A to N/P (where P is the total population). Hence, in place of Postulates I and II, we might consider the following:

POSTULATE III*a*. *The sum of the squares of the deviations of the A/a from their true values;* or

POSTULATE III*b*. *The sum of the squares of the deviations of the a/A from their true values; should be a minimum.*

It can be shown, however, that III*a* favors the smaller states even more than I*c* does, while III*b* favors the larger states even more than I*d* does. In other words, Postulates III*a* and III*b* violate, in opposite directions, all four of the conditions expressed in our Fundamental Principle. Since there is no mathematical reason for adopting either to the exclusion of the other, both should be rejected.

The same remark applies if in Postulates III*a* and III*b* the word "square" is replaced by "absolute value."[2]

Hence it is clear that if a *numerical measure of injustice* is desired, both the deviation of A/a and the deviation of a/A should be taken into account together. That is, any formula which reports A/a, say, as too large by a certain amount, should also report a/A as too small by the same amount. The formulas suggested by the simple application of the idea of least squares, as shown in the preceding paragraph, do not have this property. A combination of these formulas suggests, however, the following postulate, in which α denotes the theoretical value of a.

POSTULATE III. *In a satisfactory apportionment, the sum T of terms of the form Ae^2 where*

$$e = \frac{(a/A) - (\alpha/A)}{\sqrt{(a/A)(\alpha/A)}} = \frac{(A/\alpha) - (A/a)}{\sqrt{(A/\alpha)(A/a)}} = \frac{a - \alpha}{\sqrt{a\alpha}}$$

should be a minimum.[3]

For purposes of computation, this total error, T, may be replaced by an average error, $E = \sqrt{S/N}$, where S is the sum of terms of the form $\alpha e^2 - (a - \alpha)^2/a$.

The method determined by Postulate III is precisely the same as the method of equal proportions based on Postulates I and II.

[1] This article contains the substance of two papers presented to the American Mathematical Society, December 28, 1920, and February 26, 1921. Further details, with proofs and examples, will be published either in the Transactions of the American Mathematical Society, or in the Quarterly Publication of the American Statistical Association, or in the American Mathematical Monthly.

For the history of the subject see W. F. Willcox, "The Apportionment of Representatives" (presidential address at the annual meeting of the American Economic Association, December 1915), published in the American Economic Review, Vol. 6, No. 1, Supplement, pp. 1–16, March, 1916. See also 62d Congress, 1st Session, House of Representatives, Report No. 12, pp. 1–108, April 25, 1911, and John H. Humphreys, "Proportional Representation," London, 1911.

The most important of the methods hitherto known are four:

The *Vinton method of 1850*, long in use in Congress, is known to lead to an "Alabama paradox;" that is, an increase in the total size of the House may cause a decrease in the representation of some state.

The *Hill method of alternate ratios*, proposed by Dr. J. A. Hill in 1910 but not adopted, comes very near to satisfying the postulates of the present paper, and uses for the first time (though only partially) the idea of the geometric mean. The method is incomplete however, since it can be shown to lead to an Alabama paradox.

The *Willcox method of major fractions*, devised by Professor W. F. Willcox in 1900–1910, and now in use in Congress, employs, in effect, a working rule like ours with multipliers: Inf., 2/3, 2/5, 2/7, ...; it is essentially the same as the method of the arithmetic mean, and therefore favors the larger states unduly, just as the hitherto unsuspected but equally justifiable method of the harmonic mean favors the smaller states unduly. (It may be noted that the name "major fractions" is somewhat misleading, since the Willcox major fraction is not a major fraction of the true quota, but a major fraction of an artificial quota, scaled up or down from the true quota to meet the requirements of the computation.)

The *d'Hondt method,* originated in Belgium and now widely used in European elections, employs "multipliers" 1, 1/2, 1/3, 1/4... and can be shown to favor the larger states to the extent of violating all four of the conditions expressed in our Fundamental Principle.

[2] If one should try to minimize the *sum of the squares* (or the sum of the absolute values) *of the deviations of the a's themselves* from their true values (with or without "weighting" by the population of the state), the resulting methods would all lead to an Alabama paradox. The same is true of the weighted sum of the absolute values of the deviations of a/A (or of A/a). The same is also true of the absolute values of the logarithms of the ratios between the a'S and their true values.

[3] This Postulate III was added on April 23, after Professor F. W. Owens had shown (at the meeting of the American Mathematical Society on February 26) that the method of minimizing the sum of terms like $A[(a/A) - (\alpha/A)]^2$ leads to th same result as the Willcox method of major fractions. It may be noted that the method of minimizing the sum of terms like $a[(A/a) - (A/\alpha)]^2$ leads, not as one might expect, to the method of the harmonic mean, but to the method of the geometric mean.

CURRENT MAPS OF THE LOCATION OF THE MUTANT GENES OF DROSOPHILA MELANOGASTER[1]

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The maps that have been published[2] showing the distribution of the mutant genes of D. melanogaster can now be much improved because of the discovery of new mutants and the accumulation of crossover data. Figure 1 gives in simplified form the maps that are in use in our laboratory.

The distances on the maps are based on the total amount of crossing over between the loci, one unit of distance representing one per cent of crossing over. The map-distances are the same as the observed crossover values or "percentages of exchange" whenever the two loci considered are so close together that no, or only a negligible amount of, double crossing over occurs between them. In the first (X-) chromosome this practical equivalence of map-distance and exchange-value holds for loci not farther apart than about 15 units. In the middle of the second chromosome and of the third chromosome the equivalence holds for only about 10 units. In the end-regions of the second and third chromosomes it holds up to nearly 20 units. For distances somewhat greater than these the map-distances exceed the observed percentages of exchange by an amount equal to twice the percentage of double crossing over between the given loci. For still greater distances the difference includes also three times the percentage of triples. The number of quadruple crossovers is negligible except perhaps when the whole length of the second chromosome is to