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An Analytic Generalization of the Rogers-Ramanujan Identities for Odd Moduli

(basic hypergeometric series/partitions/combinational identities/partition identities)

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ABSTRACT A $(k-1)$-fold Eulerian series expansion is given for $\Pi(1-q^n)^{-1}$, where the product runs over all positive integers n that are not congruent to $0,i$ or $-i$ modulo $2k+1$. The Rogers-Ramanujan identities are the cases $k = i = 2$ and $k = i + 1 = 2$.

1. Introduction

The Rogers-Ramanujan identities have a long and interesting history. In 1894, L. J. Rogers proved the following two identities which have subsequently become known as the Rogers-Ramanujan or Rogers-Ramanujan-Schur identities (see ref. 1; pp. 90–99, for the early history of these identities):

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 2 \pmod 5}}^{\infty}(1-q^n)^{-1}, \qquad [1.1]$$

and

$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(q)_n} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 1 \pmod 5}}^{\infty}(1-q^n)^{-1}, \qquad [1.2]$$

where $(a)_0 = 1$, $(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, and $|q| < 1$.

The search for identities of a similar character has gone on intermittently ever since; however, no simple general identities of this nature have ever been discovered.

V. N. Singh's formula (ref. 2, Eq. 3.6) comes the closest to this objective; however it contains a rather complicated representation of the Alder polynomial.

L. J. Rogers (ref. 3, p. 331, Eq. 6) also discovered further identities related to the modulus 7; for example, he proved that

$$\prod_{m=1}^{\infty}(1+q^m)\sum_{n\geq 0}\frac{q^{2n^2}}{(q^2;q^2)_n(-q)_{2n}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 3 \pmod 7}}^{\infty}(1-q^n)^{-1}. \qquad [1.3]$$

In this same vein, W. N. Bailey (ref. 4, p. 421) proved three identities related to the modulus 9; for example

$$\prod_{m=1}^{\infty}(1+q^m+q^{2m})\sum_{m\geq 0}\frac{q^{3m^2}(q)_{3m}}{(q^3;q^3)_m(q^3;q^3)_{2m}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 4 \pmod 9}}^{\infty}(1-q^n)^{-1}. \qquad [1.4]$$

A large amount of work was done on identities of this type, and a collection of some 130 such identities was published by L. J. Slater (5). In each of the identities given by Slater, the modulus arising in the infinite product is always ≤ 64 and always has its prime factor among $\{2,3,5,7\}$.

Recently (6) five double series identities were obtained related to the modulus 11; thus

$$\prod_{m=1}^{\infty}(1+q^m+q^{2m}+q^{3m})\sum_{n\geq 0}\sum_{j\geq 0} \times \frac{q^{4n^2+12nj+8j^2+j}(-1)^j(q)_{4n+2j}}{(q^4;q^4)_n(q^2;q^2)_j(q^4;q^4)_{2n+2j}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 5 \pmod{11}}}^{\infty} \times (1-q^n)^{-1}. \qquad [1.5]$$

Unfortunately the methods used to prove [1.5] (as well as those used by Rogers, Bailey, and Slater) seem limited to special cases and do not provide general theorems for an infinite family of moduli.

In attempting to generalize fully [1.1] and [1.2], H. L. Alder (7) was able to show that there exist polynomials $G_{k,i}(n;q)$ such that

$$\sum_{n\geq 0}\frac{G_{k,i}(n;q)}{(q)_n} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty}(1-q^n)^{-1}; \qquad [1.6]$$

however neither his work nor subsequent papers (refs. 7; 8, Sec. 6; 9; 10; 2; 11; 12) on the Alder polynomials yielded analytic identities of the elegance and simplicity of [1.1] and [1.2].

The object of this paper is to prove the following generalization of the Rogers-Ramanujan identities:

THEOREM 1. Let $1 \leq i \leq k$ be integers; then

$$\sum_{n_1,n_2,\cdots,n_{k-1}\geq 0}\frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm i \pmod{2k+1}}}^{\infty}(1-q^n)^{-1}, \qquad [1.7]$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

We remark that this result reduces to [1.1] for $k = i = 2$, and to [1.2] for $k = 2$, $i = 1$; otherwise, to our knowledge, no special case of this result has appeared in the literature. We should point out that the work in Section 3 here shows that V. N. Singh's formula (ref. 2, Eq. 3.6) can be transformed into *Theorem 1* in the case $k = i$ by suitable rearrangements of series. For example, with $k = i = 3$, we obtain

$$\sum_{n,m\geq 0}^{\infty} \frac{q^{2n^2+2nm+m^2}}{(q)_n(q)_m} = \prod_{\substack{n=1\\ n\not\equiv 0,\pm 3 \pmod 7}}^{\infty} (1-q^n)^{-1} \quad [1.8]$$

instead of [1.3].

It is important to note that the technique we shall use to prove *Theorem 1* is applicable to a number of problems in the theory of partitions, and we hope in time to apply it to some of the questions raised in (ref. 13; Sec. 3) concerning partition identities.

After proving *Theorem 1* in Section 2, we shall utilize our results to obtain a new formula for the Alder polynomials.

2. Proof of Theorem 1

We begin with a function that was originally introduced by L. J. Rogers (14) and was considered independently by A. Selberg (15).

$$Q_{k,i}(x) = \sum_{n\geq 0} \times \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)-in}(1-x^i q^{(2n+1)i})}{(q)_n(xq^{n+1})_\infty} \quad [2.1]$$

where $(a)_\infty = (a;q)_\infty = \lim_{n\to\infty}(a;q)_n$.

For completeness we include a proof of the following functional equation that was independently discovered by Rogers (14) and Selberg (15).

$$Q_{k,i}(x) - Q_{k,i-1}(x) = (xq)^{i-1}Q_{k,k-i+1}(xq). \quad [2.2]$$

This follows from the fact that

$$Q_{k,i}(x) - Q_{k,i-1}(x) = \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)}}{(q)_n(xq^{n+1})_\infty}(q^{-in} - x^i q^{(n+1)i} - q^{(-i+1)n} + x^{i-1}q^{(n+1)(i-1)})$$

$$= \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)}}{(q)_n(xq^{n+1})_\infty}(q^{-in}(1-q^n) + x^{i-1}q^{(n+1)(i-1)}(1-xq^{n+1})) = \sum_{n\geq 1} \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)-in}}{(q)_{n-1}(xq^{n+1})_\infty}$$

$$+ (xq)^{i-1}\sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)+n(i-1)}}{(q)_n(xq^{n+2})_\infty} = -x^k q^{2k+1-i}\sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)+(2k+1-i)n}}{(q)_n(xq^{n+2})_\infty} + (xq)^{i-1}$$

$$\times \sum_{n\geq 0} \frac{(-1)^n x^{kn} q^{1/2(2k+1)n(n+1)+n(i-1)}}{(q)_n(xq^{n+2})_\infty} = (xq)^{i-1}\sum_{n\geq 0} \frac{(-1)^n (xq)^{kn} q^{1/2(2k+1)n(n+1)-(k-i+1)n}(1-(xq)^{k-i+1}q^{(2n+1)(k-i+1)})}{(q)_n(xq^{n+2})_\infty}$$

$$= (xq)^{i-1}Q_{k,k-i+1}(xq).$$

Since $Q_{k,0}(x) = 0$ by [2.1], we see that by [2.2]

$$Q_{1,1}(x) = Q_{1,1}(xq) = Q_{1,1}(xq^2) = \cdots$$
$$= \lim_{n\to\infty} Q_{1,1}(xq^n) = Q_{1,1}(0) = 1. \quad [2.3]$$

The next formula is new and is the key to the proof of *Theorem 1*.

$$Q_{k,i}(x) = \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n} Q_{k-1,i}(xq^{2n}), \quad [2.4]$$

for $1 \leq i \leq k$, $k \geq 2$.

To establish [2.4] we begin by observing that if we define $Q_{k,i}(x) = \sum_{M,N\geq 0} C_{k,i}(M,N)x^M q^N$, then by [2.1] and [2.2]: (*i*) $C_{k,0}(M,N) = 0$ for all M and N, (*ii*) $C_{k,i}(M,N) = 0$ if either M or N is nonpositive and $M^2 + N^2 \neq 0$, (*iii*) $C_{k,i}(0,0) = 1$ for $1 \leq i \leq k$, (*iv*) $C_{k,i}(M,N) - C_{k,i-1}(M,N) = C_{k,k-i+1}(M-i+1, N-M)$ for $1 \leq i \leq k$. Now these four conditions uniquely determine the $C_{k,i}(M,N)$; this is easily established by a double induction first on N and then on i. Hence we see that the $Q_{k,i}(x)$ $(0 \leq i \leq k)$ are the only functions analytic in x and q around (0,0) that satisfy the functional equation [2.2] for $1 \leq i \leq k$ and the boundary conditions $Q_{k,0}(x) \equiv 0$, $Q_{k,i}(0) = 1$, $1 \leq i \leq k$. To prove [2.4] then, we need only show that the right hand side of this identity also fulfills these defining conditions. We define for $k \geq 2$, $0 \leq i \leq k$,

$$R_{k,i}(x) = \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n} Q_{k-1,i}(xq^{2n}).$$

Immediately we see that $R_{k,0}(x) = 0$ and $R_{k,i}(0) = 1$ for $1 \leq i \leq k$. Furthermore for $1 \leq i \leq k$

$$R_{k,i}(x) - R_{k,i-1}(x) = \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}$$
$$\times (Q_{k-1,i}(xq^{2n}) - q^n Q_{k-1,i-1}(xq^{2n}))$$
$$= \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(Q_{k-1,i-1}(xq^{2n})$$
$$+ (xq^{2n+1})^{i-1}Q_{k-1,k-i}(xq^{2n+1}) - q^n Q_{k-1,i-1}(xq^{2n}))$$
$$= \sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k-i)n}}{(q)_n}(1-q^n)Q_{k-1,i-1}(xq^{2n})$$
$$+ (xq)^{i-1}\sum_{n\geq 0} \frac{x^{(k-1)n}q^{(k-1)n^2+(k+i-2)n}}{(q)_n}$$
$$\times (Q_{k-1,k-i+1}(xq^{2n+1}) - (xq^{2n+2})^{k-i}Q_{k-1,i-1}(xq^{2n+2}))$$
$$= \sum_{n\geq 0} \frac{x^{(k-1)n+(k-1)}q^{(k-1)n^2+(3k-i-2)n+(2k-i-1)}}{(q)_n}$$
$$\times Q_{k-1,i-1}(xq^{2n+2}) + (xq)^{i-1}R_{k,k-i+1}(xq)$$
$$- \sum_{n\geq 0} \frac{x^{(k-1)n+k-1}q^{(k-1)n^2+(3k-i-2)n+(2k-i-1)}}{(q)_n}$$
$$\times Q_{k-1,i-1}(xq^{2n+2}) = (xq)^{i-1}R_{k,k-i+1}(xq).$$

Thus, since the $R_{k,i}(x)$ fulfill all the defining conditions, we see that $R_{k,i}(x) = Q_{k,i}(x)$ for $0 \leqq i \leqq k$, $k \geqq 2$. Hence [2.4] is established

Next we wish to show that for $1 \leqq i \leqq k$, $k \geqq 2$,

$$Q_{k,i}(x) = \sum_{n_1, n_2, \cdots, n_{k-1} \geq 0} \frac{x^{N_1+N_2+\cdots+N_{k-1}} q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}, \qquad [2.5]$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$.

When $k = 2$, we see by [2.4] and [2.3] that

$$Q_{2,1}(x) = \sum_{n \geq 0} \frac{x^n q^{n^2+n}}{(q)_n},$$

which is just [2.5] for $k = 2$, $i = 1$. Furthermore by [2.2]

$$Q_{2,2}(x) = Q_{2,1}(xq^{-1}) = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n},$$

which is [2.5] for $k = i = 2$. Hence [2.5] is valid when $k = 2$.

We now assume [2.5] is valid for $k - 1$. Then for $1 \leqq i \leqq k - 1$, we see by [2.4] that

$$Q_{k,i}(x) = \sum_{n_{k-1} \geq 0} \frac{x^{(k-1)n_{k-1}} q^{(k-1)n_{k-1}^2+(k-i)n_{k-1}}}{(q)_{n_{k-1}}} \cdot \sum_{n_1, \cdots, n_{k-2} \geq 0} \frac{(xq^{2n_{k-1}})^{\nu_1+\nu_2+\cdots+\nu_{k-2}} q^{\nu_1^2+\nu_2^2+\cdots+\nu_{k-2}^2+\nu_i+\cdots+\nu_{k-2}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}},$$

where $\nu_j = n_j + n_{j+1} + \cdots + n_{k-2}$. If we write $N_j = \nu_j + n_{k-1}$, $N_{k-1} = n_{k-1}$, then

$$Q_{k,i}(x) = \sum_{n_1, \cdots, n_{k-1} \geq 0} \frac{x^{N_1+\cdots+N_{k-1}} q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}},$$

which is [2.5] for $1 \leqq i < k$. When $i = k$, by [2.2]

$$Q_{k,k}(x) = Q_{k,1}(xq^{-1}) = \sum_{n_1, \cdots, n_{k-1} \geq 0} \frac{x^{N_1+\cdots+N_{k-1}} q^{N_1^2+N_2^2+\cdots+N_{k-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}},$$

which is [2.5] for $i = k$.

We now easily deduce *Theorem 1* from [2.1] and [2.5].

$$\prod_{\substack{n=1 \\ n \not\equiv 0, \pm i \pmod{2k+1}}}^{\infty} (1 - q^n)^{-1}$$

$$= \frac{\sum_{n=0}^{\infty} (-1)^n q^{1/2(2k+1)n(n+1)-in}(1 - q^{(2n+1)i})}{(q)_\infty}$$

(by Jacobi's identity (ref. 16, pp. 169–170))

$$= Q_{k,i}(1) \qquad \text{(by [2.1])}$$

$$= \sum_{n_1, \cdots, n_{k-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$

(by [2.5]).

3. The Alder polynomials

Equation [2.5] allows us to deduce easily the following representation for the Alder polynomials, $G_{k,i}(n;q)$, defined by

$$Q_{k,i}(x) = \sum_{n \geq 0} \frac{G_{k,i}(n;q)x^n}{(q)_n}. \qquad [3.1]$$

THEOREM 2. For $1 \leqq i \leqq k$,

$$G_{k,i}(n;q) = \sum_{\substack{n_1+2n_2+\cdots+(k-1)n_{k-1}=n \\ n_1, \cdots, n_{k-1} \geq 0}} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix} \cdot \begin{bmatrix} N_{k-3} \\ n_{k-3} \end{bmatrix} \cdots \begin{bmatrix} N_1 \\ n_1 \end{bmatrix} (q^{N_1+1})_{N_2+\cdots+N_{k-1}} \cdot q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}, \qquad [3.2]$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$, and $\begin{bmatrix} m \\ n \end{bmatrix} = (q)_m(q)_n^{-1}(q)_{n-m}^{-1}$, the Gaussian polynomial.

Proof. Comparing [3.1] with [2.5], we see that

$$G_{k,i}(n;q) = (q)_n \cdot \sum_{\substack{N_1+\cdots+N_{k-1}=n \\ n_1, \cdots, n_{k-1} \geq 0}} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}}$$

$$= \sum_{\substack{n_1+2n_2+\cdots+(k-1)n_{k-1}=n \\ n_1, \cdots, n_{k-1} \geq 0}} \frac{(q)_{n_{k-2}+n_{k-1}}}{(q)_{n_{k-2}}(q)_{n_{k-1}}} \cdot \frac{(q)_{n_{k-3}+n_{k-2}+n_{k-1}}}{(q)_{n_{k-2}+n_{k-1}}(q)_{n_{k-3}}} \cdot \cdots \cdot \frac{(q)_{n_1+\cdots+n_{k-1}}}{(q)_{n_2+\cdots+n_{k-1}}(q)_{n_1}} \cdot (q^{n_1+\cdots+n_{k-1}+1})_{n_2+2n_3+\cdots+(k-2)n_{k-1}} \cdot q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}$$

$$= \sum_{\substack{n_1+2n_2+\cdots+(k-1)n_{k-1}=n \\ n_1, \cdots, n_{k-1} \geq 0}} \begin{bmatrix} N_{k-2} \\ n_{k-2} \end{bmatrix} \begin{bmatrix} N_{k-3} \\ n_{k-3} \end{bmatrix} \cdots \begin{bmatrix} N_1 \\ n_1 \end{bmatrix} \cdot (q^{N_1+1})_{N_2+\cdots+N_{k-1}} \cdot q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}, \qquad [3.3]$$

as desired.

Of all the previous representations of the Alder polynomials, [3.2] most closely resembles that of V. N. Singh (ref. 12, Eq. 3.7). This is probably due to the fact that both approaches use an iterative process. In fact, we may, by the following series of algebraic manipulations, transform Singh's representation into ours:

$$G_{k,k}(n;q) = q^{n^2} \sum_{t_1,\cdots,t_{k-2}\geqq 0} \frac{(q^{n-2t_1+1})_{2t_1}(q^{t_1-2t_2+1})_{2t_2}\cdots(q^{t_{k-3}-2t_{k-2}+1})_{2t_{k-2}}}{(q)_{t_1}(q)_{t_2}\cdots(q)_{t_{k-2}}} \cdot \frac{q^{-2t_1(n-t_1)-2t_2(t_1-t_2)-\cdots-2t_{k-2}(t_{k-3}-t_{k-2})}}{(q^{n-2t_1+1})_{t_2}(q^{t_1-2t_2+1})_{t_3}\cdots(q^{t_{k-4}-2t_{k-3}+1})_{t_{k-2}}}$$

$$= q^{n^2} \sum_{t_1,\cdots,t_{k-2}\geqq 0} \frac{(q)_n}{(q)_{n-2t_1}} \cdot \frac{(q)_{t_1}}{(q)_{t_1-2t_2}} \cdot \cdots \cdot \frac{(q)_{t_{k-3}}}{(q)_{t_{k-3}-2t_{k-2}}(q)_{t_1}(q)_{t_2}\cdots(q)_{t_{k-2}}} \cdot \frac{q^{-2t_1(n-t_1)-2t_2(t_1-t_2)-2t_3(t_2-t_3)-\cdots-2t_{k-2}(t_{k-3}-t_{k-2})}}{(q^{n-2t_1+1})_{t_2}(q^{t_1-2t_2+1})_{t_3}\cdots(q^{t_{k-4}-2t_{k-3}+1})_{t_{k-2}}}$$

$$= q^{n^2} \sum_{t_1,\cdots,t_{k-2}\geqq 0} \frac{(q)_n q^{-2t_1(n-t_1)-2t_2(t_1-t_2)-\cdots-2t_{k-2}(t_{k-3}-t_{k-2})}}{(q)_{n-2t_1+t_2}(q)_{t_1-2t_2+t_3}\cdots(q)_{t_{k-4}-2t_{k-3}+t_{k-2}}(q)_{t_{k-3}-2t_{k-2}}(q)_{t_{k-2}}}. \quad [3.4]$$

Now we let $t_{k-2} = n_{k-1}$, $t_{k-3} = n_{k-2} + 2n_{k-1}, \cdots,$ $t_{k-j-1} = n_{k-j} + 2n_{k-j+1} + \cdots + jn_{k-1}$ for $1 \leqq j \leqq k-2$. Thus $t_{k-2} = n_{k-1}$, $t_{k-3} - 2t_{k-2} = n_{k-2}$, $t_{k-j-1} - 2t_{k-j} + t_{k-j+1} = n_{k-j}$, and $t_{k-j-1} - t_{k-j} = n_{k+j} + \cdots + n_{k-j} \equiv N_{k-j}$. Therefore,

$$\begin{aligned} &n^2 - 2t_1(n-t_1) - 2t_2(t_1-t_2) - \cdots \\ &\quad - 2t_{k-2}(t_{k-3}-t_{k-2}) \\ &= n^2 - 2(N_2 + \cdots + N_{k-1})(n - N_2 - \cdots - N_{k-1}) \\ &\quad - 2(N_3 + \cdots + N_{k-1})N_2 \\ &\quad - 2(N_4 + \cdots + N_{k-1})N_3 \\ &\quad \vdots \\ &\quad - 2N_{k-1} \cdot N_{k-2} \\ &= n^2 - 2(N_2 + \cdots + N_{k-1})n + 2N_2^2 + \cdots \\ &\quad + 2N^2_{k-1} + 2\sum_{2\leqq i<j\leqq k-1} N_iN_j \\ &= (n - N_2 - \cdots - N_{k-1})^2 + N_2^2 + \cdots + N^2_{k-1}. \end{aligned}$$

Hence the above shifts in summation (i.e., $t_{k-j-1} = \sum_{l=1}^{j} ln_{k-j-1+l}$) transform [3.4] into [3.3].

4. Conclusion

Apart from the fact that we now have a reasonable analytic generalization of the Rogers-Ramanujan identities, there is at least one other significant feature related to the present work. Namely, we now have good reason to hope that the iterative use of q-difference equations (such as was used to prove [2.4]) will be instrumental in providing at least a partial answer to Question 2 of (ref. 13, Sec. 3):

"What finite linear q-difference equations with polynomial coefficients···have solutions that can be represented by 'higher dimentional' q-series?"

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