Limits of zeroes of recursively defined polynomials
(linear homogeneous recursion/chromatic polynomial)

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ABSTRACT Let \( P_n(x) \) be a sequence of polynomials satisfying a linear homogeneous recursion whose coefficients are polynomials in \( x \). Necessary and sufficient conditions are found, subject to mild nondegeneracy conditions, that a number \( x \) be a limit of zeroes of \( P_n \) in the sense that there is a sequence \( \{z_n\} \) with \( P_n(z_n) = 0, z_n \to x \). An application is given to a family of polynomials arising in a map-coloring problem.

1. Main result
Let \( P_n(x) \) be a sequence of polynomials satisfying

\[ P_{n+k}(x) = -\sum_{j=1}^{k} f_j(x) P_{n+k-j}(x). \]

where the \( f_j \) are polynomials.

The complex number \( x \) is said to be a limit of zeroes of \( P_n \) if there is a sequence \( \{z_n\} \) such that \( P_n(z_n) = 0 \) and \( z_n \to x \). Our main result is a necessary and sufficient condition, subject to two mild nondegeneracy conditions, that \( x \) be a limit of zeroes of \( P_n \).

For fixed \( x \), the solution of [1] given \( P_0(x), \ldots, P_{k-1}(x) \) depends on the roots of the characteristic equation

\[ \lambda^k + \sum_{j=1}^{k} f_j(x) \lambda^{k-j} = 0. \]

The roots, \( \lambda_j(x), j = 1, \ldots, k \), are algebraic functions of \( x \), and for any \( x \) such that the \( \lambda_j(x) \) are distinct,

\[ P_n(x) = \sum_{j=1}^{k} \alpha_j(x) \lambda_j(x)^n. \]

where the \( \alpha_j \) are determined by solving the system of equations obtained by letting \( n = 0,1,\ldots,k-1 \) in [2].

The nondegeneracy conditions needed for our result are

A. \( P_n \) satisfies no recursion of order less than \( k \).
B. There does not exist a constant \( \omega \) with \( |\omega| = 1 \) and \( \lambda_i(z) = \omega \lambda_j(z) \) for some \( i \neq j \).

THEOREM. Suppose that \( P_n \) satisfies [1], A and B. Then \( x \) is a limit of zeroes of \( P_n \) if and only if the roots can be numbered so that one of the following holds:

(i) \( |\lambda_1(x)| > |\lambda_j(x)|, 2 \leq j \leq k \), and \( \alpha_i(x) = 0 \)
(ii) \( |\lambda_1(x)| = |\lambda_2(x)| = \ldots = |\lambda_l(x)| > |\lambda_j(x)|, 1 \leq l \leq k \),

2. Comments
A bare outline of the proof, details of which will appear elsewhere, is as follows. Suppose, typically, that \( x \) is such that the \( \lambda_j(x) \) are distinct, so that the same is true in a neighborhood of \( x \). Referring to [2], it is routine to show that if \( x \) is a limit of zeroes and (ii) fails to hold, then (i) must hold. The proof of sufficiency involves showing that if (i) or (ii) holds, then in any neighborhood of \( x, Q_n(z) = P_n(z)/\lambda_1(z) \) and force \( P_n(z) \) has a zero for all sufficiently large \( n \). In each case, a winding number argument is used; if (i) holds, Rouche's theorem suffices, while if (ii) holds, a more complicated method is needed. If the \( \lambda_j(x) \) are not distinct, complications arise which are essentially technical.

If the condition A does not hold, the Theorem can be applied to the unique lowest-order recursion satisfied by \( P_n \). The situation is more interesting when B fails to hold. Variants of (i) and (ii) exist such that a limit of zeroes must satisfy one or the other. As for the question of the sufficiency of these variants, suffice it to say that if \( \lambda_1(z) = 1, \lambda_2(z) = \omega = e^{2\pi i/6} \) and the other \( \lambda_j \) satisfy B, then the answer depends on whether \( \theta \) is integral, rational but nonintegral, or irrational. Moreover, examples show that in the last case the answer seems to depend upon the degree of transcendent of \( \theta \).

3. An application
Given a map \( M \), a coloring of \( M \) is an assignment of a color to each region of \( M \) such that contiguous regions are colored differently. For a positive integer \( m \), let \( Q(M,m) \) be the number of ways \( M \) can be colored with \( m \) or fewer colors, regarding as distinct even those colorings which can be obtained from each other by permutations of the colors. G. D. Birkhoff showed (ref. 1) that there is a polynomial \( P_n \) such that \( Q(M,m) = P_n(m) \) for all positive integers \( m \). A polynomial arising in this way is called a chromatic polynomial; the four-color conjecture amounts to the nonexistence of a chromatic polynomial with a zero at 4.

In a paper submitted to the Journal of Combinatorial Theory, the first two authors consider a sequence of maps \( M_n \) consisting of an inner region and an outer region separated by \( n \) rings, each containing four regions. It is found that the corresponding sequence \( P_n \) of chromatic polynomials satisfies a recursion of the form [1], with \( k = 2 \),

\[ f_1(z) = -(z - 3)(z^3 - 9z^2 + 8z - 48), \]

\[ f_2(z) = 2(z - 3)(z^2 - 3x^2 - 5z + 5). \]

When \( z = 4 \), the characteristic equation has 2 as a repeated root, implying in particular that (ii) holds so that 4 is a limit of zeroes of \( P_n \). (Moreover, the only other real numbers which are limits of zeroes of \( P_n \) are 0, 1, 2, 3, and \( \frac{1}{2} (3 + \sqrt{5}) \).)

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