The disjunction property implies the numerical existence property

(intuitionism/arithmetic/formal systems/self reference)

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ABSTRACT   Any recursively enumerable extension of intuitionistic arithmetic which obeys the disjunction property obeys the numerical existence property. Any recursively enumerable extension of intuitionistic arithmetic proves its own disjunction property if and only if it proves its own inconsistency.

Let HA be the usual formulation of Heyting arithmetic, in the one-sorted intuitionistic predicate calculus with identity, in which function symbols for primitive recursive functions are included. Let HA_0 be the same as HA, except that the induction scheme is restricted to quantifier free formulae. A theory T, formulated in many-sorted intuitionistic predicate calculus without identity, is said to be *an extension of arithmetic* if its axioms include the axioms of HA_0.

An extension, T of arithmetic is said to have the *disjunction property* just in case for each closed consequence $A \lor B$ of T, either A is a consequence of T or B is a consequence of T. We say that T has the *numerical existence property* just in case for each closed consequence $(\exists x)(C(x))$ of T in which x is a numerical variable, there is a natural number n such that $C(\bar{n})$ is a consequence of T.

The disjunction property and the numerical existence property have been established for a variety of extensions of arithmetic, including HA itself, theories of functionals, second order arithmetic, the theory of types, and set theories. The methods used in establishing such results include cut elimination, normalization, Kripke models, Kleene's realizabilities, and Kleene's $|$. We refer the reader to ref. 1 for a survey and bibliography, and to refs. 1–6 for proofs of the fundamental result that HA obeys the numerical existence property.

In each and every case in which a method established the disjunction property for a particular extension T of arithmetic, that same method actually established the numerical existence property for T. We sought an explanation for this phenomenon.

Our discovery is that *any* extension T of arithmetic that obeys the disjunction property obeys the numerical existence property, provided the set of axioms of T is recursively enumerable. This is our principal result, and the proof is presented in detail. (The converse is of course trivial.)

In some extensions of arithmetic, such as the two-sorted intuitionistic second order arithmetic, stronger formulations of the disjunction and numerical existence properties hold, involving formulae with free variables. A generalization of our result can be obtained straight forwardly as a consequence of the result proved here, covering such formulations.

We remark that the situation is quite different with respect to non-numerical existence properties. For example, consider the theory $\{(\exists x)(R(x))\}$, where R is atomic, in the one-sorted intuitionistic predicate calculus with or without identity.

This theory obeys the (open) disjunction property, yet for no closed (open) term t is $R(t)$ derivable.

All known proofs of our principal result are based on certain self referential sentences. Our original proof was discovered in June, 1974, and Daniel Leivant and Per Lindstrom sent us two different variants of our proof whose details are, in one way or another, simpler than those in our original proof. The proof that we present here follows Leivant's informal communication, "Friedman's proof that disjunction-instantiation implies existence-instantiation," of August 8, 1974.

Until the completion of *Theorem 1*, assume that T is a recursively enumerable extension of arithmetic that obeys the disjunction property, and x,y,z,w are distinct numerical variables of T.

LEMMA 1.   *There is a one-one function $\#$ from the set of formulae of T into ω, and primitive recursive functions neg, prf, sub from ω into ω, such that (1) for all formulae A, $neg(\#(A)) = \#(\sim A)$; (2) for all formulae B, $T \vdash B$ if and only if $(\exists n)(prf(n, \#(B)) = 0$; (3) for all formulae $C = C(x)$: $sub(\#(C)) = \#(C(\overline{\#(C)}))$.*

Proof: Standard coding.

LEMMA 2.   *For each n-ary primitive recursive function symbol F of T there is a primitive recursive function $|F|$, such that $|F|(n) = m \to T \vdash F(\bar{n}) = \bar{m}$. For each primitive recursive function f there is a primitive recursive function symbol F with $f = |F|$.*

Proof: Standard.

Now fix Neg, Prf, and Sub so that $|Neg| = neg$, $|Prf| = prf$, $|Sub| = sub$.

LEMMA 3.   *Let $A(x)$ be a formula of T. Then there is a k such that $\#(A(Sub(\bar{k}))) = sub(k)$.*

Proof: Let $k = \#(A(Sub(x)))$. Then $sub(k) = \#(A(Sub(\bar{k})))$, by *Lemma 1*.

Let $+$ be the natural function symbol such that $|+| =$ addition. Take $x \leq y$ to be $(\exists z)(x + z = y)$, $x < y$ to be $x \leq y \;\&\; \sim x = y$.

We now let $P(y)$ be a formula with no free variable other then y. Let $A(x)$ be the formula $(\exists y)((Prf(y, Neg(x)) = 0 \lor P(y)) \;\&\; (\forall z)(Prf(z,x) = 0 \to y \leq z))$. By *Lemma 3*, choose k such that $\#(A(Sub(\bar{k}))) = sub(k)$.

LEMMA 4.   *If $T \vdash A(Sub(\bar{k}))$ then $T \vdash P(n)$, for some n.*

Proof: Assume $T \vdash A(Sub(\bar{k}))$. Let $prf(n, sub(k)) = 0$. Then $T \vdash Prf(\bar{n}, Sub(\bar{k})) = 0$. Now $T \vdash (\exists y)((Prf(y, Neg(Sub(\bar{k}))) = 0 \lor P(y)) \;\&\; (Prf(\bar{n}, Sub(\bar{k})) = 0 \to y \leq \bar{n}))$. Hence $T \vdash (\exists y)((Prf(y, Neg(Sub(\bar{k}))) = 0 \lor P(y)) \;\&\; y \leq \bar{n})$.

If $(\exists y \leq n)(\mathrm{prf}(y,\mathrm{neg}(\mathrm{sub}(k))) = 0)$, then $T \vdash \sim A(\mathrm{Sub}(\bar{k}))$, and hence T is inconsistent. If $\sim(\exists y \leq n)\ (\mathrm{prf}(y,\mathrm{neg}(\mathrm{sub}(k))) = 0)$, then $T \vdash \sim(\exists y)(y \leq \bar{n}\ \&\ \mathrm{Prf}(y,\mathrm{Neg}(\mathrm{Sub}(\bar{k}))) = 0)$, and hence $T \vdash (\exists y)(y \leq \bar{n}\ \&\ P(y))$. Therefore, $T \vdash \bigvee_{i \leq n} P(\bar{\imath})$. Since T has the disjunction property, clearly $T \vdash P(\bar{\imath})$ for some $i \leq n$. In either case, $T \vdash P(\bar{n})$ for some n.

LEMMA 5. *If* $T \vdash \sim A(Sub(\bar{k}))$ *then* T *is inconsistent.*

Proof: Assume $T \vdash \sim A(\mathrm{Sub}(\bar{k}))$. Let $\mathrm{prf}(n,\mathrm{neg}(\mathrm{sub}(k))) = 0$. Then $T \vdash \mathrm{Prf}(\bar{n},\mathrm{Neg}(\mathrm{Sub}(\bar{k}))) = 0$. If $(\forall z)\text{-}(\mathrm{prf}(z,\mathrm{sub}(k)) = 0 \to n \leq z)$, then $T \vdash (\forall z)(\mathrm{Prf}(x,\mathrm{Sub}(\bar{k})) = 0 \to \bar{n} \leq z)$, and so $T \vdash A(\mathrm{Sub}(\bar{k}))$. Hence T is inconsistent. If $(\exists z)(z < n\ \&\ \mathrm{prf}(z,\mathrm{sub}(k)) = 0)$, then $T \vdash A(\mathrm{Sub}(\bar{k}))$, and so again T is inconsistent.

LEMMA 6. $T \vdash (\exists y)(P(y)) \to (A(Sub(\bar{k})) \vee (\exists z)(Prf(z,\text{-}Sub(\bar{k})) = 0))$. *If* P *is primitive recursive, then* $T \vdash (\exists y)(P(y)) \to (A(Sub(\bar{k})) \vee \sim A(Sub(\bar{k})))$.

Proof: Use axioms of T. Suppose $(\exists y)(P(y))$, and let $P(y)$. If $(\forall z)(\mathrm{Prf}(z,\mathrm{Sub}(\bar{k})) = 0 \to y \leq z)$, then $A(\mathrm{Sub}(\bar{k}))$. If not, then $(\exists z)(\mathrm{Prf}(z,\mathrm{Sub}(\bar{k})) = 0)$. If P is primitive recursive, then so is $\mathrm{Prf}(y,\mathrm{Neg}(\mathrm{Sub}(\bar{k}))) = 0 \vee P(y)$. Let y be chosen least such that this holds. Hence $(\mathrm{Prf}(y,\mathrm{Neg}(\mathrm{Sub}(\bar{k}))) = 0 \vee P(y))\ \&\ (\forall z)((\mathrm{Prf}(z,\mathrm{Neg}(\mathrm{Sub}(\bar{k}))) = 0 \vee P(z)) \to y \leq z)$. We have $A(\mathrm{Sub}(\bar{k})) \leftrightarrow (\forall z)(\mathrm{Prf}(z,\mathrm{Sub}(\bar{k})) = 0 \to y \leq z)$. Therefore, $A(\mathrm{Sub}(\bar{k})) \vee \sim A(\mathrm{Sub}(\bar{k}))$.

LEMMA 7. *If* $T \vdash (\exists y)(P(y))$ *and* P *is primitive recursive, then for some* n, $T \vdash P(\bar{n})$.

Proof: By *Lemma 6* and the disjunction property, the hypotheses yield $T \vdash A(\mathrm{Sub}(\bar{k}))$ or $T \vdash \sim A(\mathrm{Sub}(\bar{k}))$. The conclusion follows by *Lemmas 4* and *5*.

LEMMA 8. *If* $T \vdash (\exists y)(P(y))$ *then for some* n, $T \vdash P(\bar{n})$.

Proof: If $T \vdash (\exists y)(P(y))$ then $T \vdash (A(\mathrm{Sub}(\bar{k})) \vee (\exists z)(\mathrm{Prf}(z,\mathrm{Sub}(\bar{k})) = 0))$, by *Lemma 6*. By the disjunction property, $T \vdash A(\mathrm{Sub}(\bar{k}))$ or $T \vdash (\exists z)(\mathrm{Prf}(z,\mathrm{Sub}(\bar{k})) = 0)$. If the former holds, we are done by *Lemma 4*. If the latter holds, then by *Lemma 7*, let $T \vdash \mathrm{Prf}(\bar{n},\mathrm{Sub}(\bar{k})) = 0$. If $\mathrm{prf}(n,\mathrm{sub}(k)) \neq 0$, then T is inconsistent. If $\mathrm{prf}(n,\mathrm{sub}(k)) = 0$, then $T \vdash A(\mathrm{Sub}(\bar{k}))$. Hence in either case, $T \vdash A(\mathrm{Sub}(\bar{k}))$. The conclusion again follows by *Lemma 4*.

We have now shown the following:

THEOREM 1. *Every recursively enumerable extension of arithmetic which obeys the disjunction property, also obeys the numerical existence property.*

We remark that the proof of *Theorem 1* can be formalized in HA_0.

In *Theorem 1*, the recursive enumerability of T is needed, since it can be shown that there are Δ^0_2 extensions of arithmetic that obey the disjunction property, but not the numerical existence property.

In ref. 7 it is shown that no consistent recursively enumerable extension of arithmetic that obeys the disjunction property proves its own disjunction property (where the latter is formulated in terms of an adequate proof predicate

for the extension)*. In *Theorem 2* below, we obtain this result as an immediate consequence of *Theorem 1*, and our result that any recursively enumerable extension of arithmetic that proves its own disjunction property proves its own inconsistency. This latter result is apparently not obtainable by the methods of ref. 7, and we obtain it as an easy consequence of *Theorem 1* by means of the following lemma, which is known as Löb's theorem (see ref. 8).

LEMMA 9. *Let* T *be a recursively axiomatized extension of* HA, *and let* '$T \vdash$' *be adequately expressed in* T. *Suppose* A *is a sentence such that* $T \vdash ((T \vdash A) \to A)$. *Then* $T \vdash A$.

Proof: Assume hypotheses. By self reference, let $T \vdash (B \leftrightarrow T \vdash (B \to A))$, where B is a sentence. Note that $T \vdash (B \to T \vdash B)$. Hence $T \vdash (B \to T \vdash A)$. By hypothesis, $T \vdash B \to A$. Hence $T \vdash B$. So $T \vdash A$.

THEOREM 2. *Let* T *be a recursively enumerable extension of arithmetic, and let* '$T \vdash$' *be adequately expressed in* T. *Suppose that for all sentences* A,B *we have* $T \vdash ((T \vdash A \vee B) \to T \vdash A \vee T \vdash B)$. *Then* $T \vdash (T \vdash 1 = 0)$. *Furthermore if, additionally,* T *obeys the disjunction property, then* $T \vdash 1 = 0$.

Proof: Since the proof of *Theorem 1* can be formalized in HA_0, we see that each instance of the numerical existence property (for T) is provable in T. Hence $T \vdash ((T \vdash (T \vdash 1 = 0)) \to (\exists x)(T \vdash "\bar{x}$ is an index of a proof in T of $1 = 0"))$. Note that $T \vdash ((\exists x)(T \vdash "\bar{x}$ is an index of a proof in T of $1 = 0") \to T \vdash 1 = 0)$. Hence $T \vdash ((T \vdash (T \vdash 1 = 0)) \to T \vdash 1 = 0)$. By *Lemma 9*, $T \vdash (T \vdash 1 = 0)$.

If, in addition, T obeys the disjunction property, then by *Theorem 1*, T obeys the existence property. Since $T \vdash (T \vdash 1 = 0)$, we have $(\exists x)(T \vdash \bar{x}$ is an index of a proof in T of $1 = 0)$, and hence $T \vdash 1 = 0$.

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* In ref. 7, the (needed) assumption that T has the disjunction property is used in the proofs of Corollaries 1 and 3, although it is not mentioned in the hypotheses.