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On a sharpened form of the Schauder fixed-point theorem

(compact mappings/inward and outward sets)

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ABSTRACT **If K is a compact convex subset of a locally convex topological vector space X, we consider a continuous mapping f of K into X. A fixed-point theorem is proved for such a map f under the assumption that for a given continuous real-valued function p on $K \times X$ with $p(x,y)$ convex in y and for each point x in K not fixed by f, there exists a point y in the inward set $I_K(x)$ generated by K at x with $p(x,y - f(x))$ less than $p(x,x - f(x))$. For X a Banach space, in particular, this yields a sharp extension and a drastic simplification of the fixed point theory of weakly inward (and weakly outward) mappings. The result comes close in the domain of mappings of compact convex sets to the thrust of fixed point conditions of the Leray–Schauder type for compact maps of sets with interior in X.**

Let X be a Banach space, K a compact convex subset of X. The classical Schauder fixed-point theorem, which is one of the basic tools in dealing with nonlinear problems in analysis, asserts that each continuous mapping f of K into K has a fixed point. An equivalent form of the theorem states that if C is a closed convex subset of X and g is a continuous self-mapping of C with $g(C)$ relatively compact, then g has a fixed point. If one considers a more general situation in which the mappings f and g map K and C, respectively, not into themselves but into the containing space X, a curious anomaly arises. If C has a nonempty interior, the theory of the Leray–Schauder degree yields the conclusion that (even with C nonconvex) if for each point on the boundary of C, $g(x) \neq rx$ for any $r \geq 1$, then g has a fixed point in C. No full extension of this type of result for compact K has been established, though a partial generalization is contained in the theory of inward and outward mappings.

For each point x of K, let the inward and outward sets of K at x, $I_K(x)$, and $O_K(x)$ be defined by

$$I_K(x) = \{y \mid y\ X; \text{there exist } u \text{ in } K \text{ and } r > 0 \text{ such that } y = x + r(u - x)\}$$

and

$$O_K(x) = \{y \mid y\ X; \text{there exist } u \text{ in } K \text{ and } r > 0 \text{ such that } y = x - r(u - x)\}.$$

A mapping f of K into X is said to be inward if for each x in K, $f(x)$ lies in $I_K(x)$, and is said to be outward if each $f(x)$ lies in $O_K(x)$. More generally, f is said to be weakly inward if each $f(x)$ lies in the closure of $I_K(x)$ and weakly outward if each $f(x)$ lies in the closure of $O_K(x)$.

The definition and study of inward and outward mappings was begun by B. Halpern in his (unpublished) University of California, Los Angeles, doctoral dissertation of 1965 in which he proved that if X is a strictly convex Banach space (or more generally, semi-strictly convex), then each inward and outward mapping of a compact convex K has a fixed point. Using a different method of argument based upon a systematic application of fixed point theorems for multi-valued mappings, the writer (1, 2) extended this result to upper-semi continuous multi-valued mappings in arbitrary Banach spaces and locally convex topological vector spaces. Halpern and Bergman (3) established the fixed-point property for single-valued weakly inward and outward mappings in general Banach spaces, while Halpern (4) extended the writer's method to multi-valued weakly inward and outward mappings. Fan (5) gave some interesting technical sharpenings of the writer's method, while a number of recent investigators [e.g., Reich (6), Fitzpatrick and Petryshyn (7), and Caristi (8)] have studied extensions to noncompact mappings.

It is our purpose here to establish the following rather sharp improvement of these results for single-valued mappings.

THEOREM 1. *Let* X *be a locally convex topological vector space,* K *a compact convex subset of* X, f *a continuous mapping of* K *into* X. *Suppose that* p *is a continuous nonnegative real-valued function on* K × X *such that for all* x *in* K, p(x,·) *is a convex function on* X. *Suppose that for each* x *in* K *such that* x ≠ f(x), *there exists a point* y *in* $I_K(x)$ *such that* p(x,y − f(x)) < p(x,x − f(x)).

Then f *has a fixed point in* K.

COROLLARY 1. *Let* X *be a Banach space,* K *a compact convex subset of* X, f *a continuous map of* K *into* X. *Suppose that for each* x *in* K *with* x ≠ f(x), *there exists* y *in* $I_K(x)$ *such that* ‖y − f(x)‖ < ‖x − f(x)‖.

Then f *has a fixed point.*

Corollary 1 is obtained by setting $p(x,y) = \|y\|$.

COROLLARY 2. *Let* X *be a Banach space,* K *a compact convex subset of* X, f *a continuous map of* K *into* X *such that for each* x *in* K, f(x) *lies in the closure of* $I_K(x)$. *Then* f *has a fixed point.*

Proof of Corollary 2: For any x in K such that $x \neq f(x)$, $\|x - f(x)\| > 0$. Because we can make $\|y - f(x)\|$ arbitrarily small by choosing y in $I_K(x)$, we can in particular choose such y so that $\|y - f(x)\| < \|x - f(x)\|$.

Another corollary of *Theorem 1* is the corresponding result in which $I_K(x)$ is replaced by $O_K(x)$, namely:

THEOREM 2. *Let* X *be a locally convex topological vector space,* K *a compact convex subset of* X, f *a continuous mapping of* K *into* X. *Let* p *be as in* Theorem 1. *Suppose that for each* x *in* K *for which* x ≠ f(x), *there exists* y *in* $O_K(x)$ *such that*

$$p(x,y - f(x)) < p(x,x - f(x)).$$

Then f *has a fixed point.*

Proof of Theorem 2 from Theorem 1: Let g be the mapping of K into X given by $g(x) = 2x - f(x)$. Then $x - g(x) = -[x - f(x)]$, so that f and g have the same fixed points.

For any y in $O_K(x)$, let $z = 2x - y$. Then z lies in $I_K(x)$. Set $q(x,y) = p(x, -y)$ for x in K, y in X. Then the real-valued

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function q on $K \times X$ satisfies the same conditions as p. Moreover

$$q(x,z - g(x)) = q(x, -(y - f(x)) = p(x,y - f(x))$$

and if

$$p(x,y - f(x)) < p(x,x - f(x)),$$

we have

$$q(x,z - g(x)) < p(x,x - f(x)) = p(x, -(x - g(x)) = q(x,x - g(x)).$$

Applying the conclusion of *Theorem 1* to the mapping g, we see that g has a fixed point. Hence, f has a fixed point.

COROLLARY 1′. *Let* X *be a Banach space,* K *a compact convex subset of* X, f *a continuous map of* K *into* X. *Suppose that for each* x *in* K *with* $x \neq f(x)$, *there exists* y *in* $O_K(x)$ *such that* $\|y - f(x)\| < \|x - f(x)\|$. *Then* f *has a fixed point.*

COROLLARY 2′. *Let* X *be a Banach space,* K *a compact convex subset of* X, f *a continuous map of* K *into* X *such that for each* x *in* K, f(x) *lies in the closure of* $O_K(x)$. *Then* f *has a fixed point in* K.

To prove *Theorem 1* we shall apply an elementary but very basic fixed point theorem for multi-valued mappings due to Fan (9), reproved and applied to inward and outward mappings by the writer (2):

PROPOSITION 1. *Let* K *be a compact convex subset of a topological vector space* E, S *a map of* K *into* 2^K. *Suppose that for each* x *in* K, S(x) *is nonempty and convex, while for each* u *in* K, $S^{-1}(u)$ *is open. Then there exists* x_0 *in* K *such that* $x_0 \in S(x_0)$.

Proof of Proposition 1: Because K is compact and covered by the family of open sets $\{S^{-1}(u): u \in K\}$, there exists a finite subcovering $\{S^{-1}(u_j): j = 1, \ldots, r\}$ and a corresponding partition of unity $\{\alpha_j(x)\}$. We form the mapping q of K into a finite-dimensional closed convex subset of K by setting

$$q(x) = \sum_{j=1}^{r} \alpha_j(x)u_j.$$

For each j such that $\alpha_j(x) \neq 0$, u_j lies in $S(x)$. Because $S(x)$ is convex, it follows that for each x, $q(x)$ lies in $S(x)$. By Brouwer's fixed-point theorem, there exists x_0 in K such that $x_0 = q(x_0)$. For this x_0, $x_0 \in S(x_0)$. q.e.d.

From *Proposition 1,* we obtain an immediate proof of the following fixed-point theorem:

Proposition 2: Let E *be a locally convex topological vector space,* K *a compact convex subset of* E, f *a continuous mapping of* K *into* E. *Let* p *be a continuous mapping of* K × E *into the reals such that for each real number* r *and for each* x *in* K, $\{y | y \in E, p(x,y) < r\}$ *is convex. Suppose that for each* x *in* K *for which* $x \neq f(x)$, *there exists* y *in* K *such that* $p(x,y - f(x)) < p(x,x - f(x))$. *Then* f *has a fixed point in* K.

Proof of Proposition 2: Suppose that f has no fixed points in K. For each x in K, we let $S(x) = \{y | y \in K, p(x,y - f(x)) < p(x,x - f(x))\}$. For each x in K, $S(x)$ is non-empty and convex by hypothesis. Because p is continuous, for each u in K, $S^{-1}(y)$ is open. By *Proposition 1,* S has a fixed point, which contradicts its definition. q.e.d.

Proposition 2 is a slight generalization of a result of Fan (5).

Proof of Theorem 1 from Proposition 2: For each x in K such that $x \neq f(x)$, there exists y in $I_K(x)$ such that $p(x,y - f(x)) < p(x,x - f(x))$. If y lies in K, the hypothesis of *Proposition 1* is valid for that x. If y lies outside of K, then there exists u in K such that y lies on the ray from x through u. In particular, there exists β with $0 < \beta < 1$ such that $u = (1 - \beta)x + \beta y$. By the convexity of $p(x,\cdot)$, however,

$$p(x,u - f(x)) = p(x,(1 - \beta)(x - f(x)) + \beta(y - f(x))) \leq (1 - \beta)p(x,x - f(x)) + \beta p(x,y - f(x)) < p(x,x - f(x)).$$

Hence, the hypothesis of *Proposition 1* is fulfilled for each point and f has a fixed point in K. q.e.d.

An interesting variant of *Theorems 1* and *2* may be derived by employing the concept of the subgradient of a convex function. We recall that if h is a convex real-valued function on the space X, its subgradient is the mapping ∂h from X to 2^{X^*}, [in which X^* is the space of continuous linear functionals on X and (w,u) denotes the pairing between the functional w and the element u] given by:

$$(\partial h)(x) = \{w | w \in X^*, \text{ for all } y \text{ in } X, h(y) \geq h(x) + (w,y - x)\}.$$

THEOREM 3. *Let* X *be a locally convex topological vector space,* K *a compact convex subset of* X, f *a continuous mapping of* K *into* X. *Suppose that* p *is a continuous real-valued function on* K × X, *such that if we set* $p_x(y) = p(x,y)$, *then each* p_x *is a convex function on* X. *Suppose that* $(\partial p_x)(u)$ *is nonempty for each* x *in* K, u *in* X, *and that for each* x *in* K *which is not fixed by* f, *there exists a point* y *in* $I_K(x)$ *such that for all* w *in* $(\partial p_x)(x - f(x))$,

$$(w,y - x) < 0.$$

Then f *has a fixed point in* K.

Proof of Theorem 3: We reduce the conclusion of *Theorem 3* to that of *Theorem 1* by showing that the hypothesis of *Theorem 1* must be satisfied. Suppose the contrary. Then there exists a point x in K such that for all y in $I_K(x)$, $p_x(x - f(x)) \leq p_x(y - f(x))$. Fix any such y, and for $t < 0$, let $y_y = (1 - t)x + ty$. Then

$$p_x(x - f(x)) \leq p_x(y_t - f(x)).$$

By assumption, there exists w_t in $(\partial p_x)(y_t - f(x))$. For such w_t, we know by the definition of subgradient that

$$p_x(y_t - f(x)) + (w_t,x - y_t) \leq p_x(x - f(x)).$$

Combining this with the preceding inequality, we find:

$$0 \leq (w_t,y_t - x) = t(w_t,y - x),$$

and because we have chosen $t > 0$, $(w_t,y - x) \geqslant 0$. As t goes to 0, the family $\{w_t\}$ lies in a subset of X^* that is compact with respect to the weak topology on X^* induced by the pairing with X. Any weak accumulation point w must lie in $(\partial p_x)(x - f(x))$. For such w, we must have $(w,y - x) \geq 0$. This last inequality contradicts the hypothesis of *Theorem 3,* however, and the proof is complete. q.e.d.

A similar argument using *Theorem 2* leads to the following dual result:

THEOREM 4. *Let* X *be a locally convex topological vector space,* K *a compact convex subset of* X, f *a continuous mapping of* K *into* X. *Suppose that* p *is a continuous real-valued function on* K × X *such that if we set* $p_x(y) = p(x,y)$, *each* p_x *is a convex function on* X. *Suppose that for each* x *in* K, $(\partial p_x)(u)$ *is non-empty for each* u *in* X. *Suppose further that for each* x *in* K *not fixed by* f, *there exists a point* y *in* $I_K(x)$ *such that for all* w *in* $(\partial p_x)[x - f(x)]$,

$$(w,y - x) > 0.$$

Then f *has a fixed point in* K.

Remark on the proof of *Theorem 4:* We note that if y lies in $I_K(x)$, then $z = 2x - y$ lies in $O_K(x)$, and conversely. Moreover, $(w,y - x) = -(w,z - x)$. Hence the hypothesis of *Theorem 4* implies that $(w,z - x) < 0$. The remainder of the proof follows the pattern of proof of *Theorem 3,* if we replace the application of *Theorem 1* by that of *Theorem 2.*

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