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Analytic torsion and Reidemeister torsion

(Riemannian manifold/Laplacian/boundary conditions/Sobolev constant)

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ABSTRACT We announce a proof of the conjecture of Ray and Singer that for a compact Riemannian manifold the analytic torsion and Reidemeister torsion are equal. The proof involves studying the heat equation for certain manifolds **M**, equipped with metrics g_u, $0 < u < 1$ which degenerate in a prescribed way at the boundary ∂**M**, as $u \to 0,1$.

1. Introduction

Let K^* be the real cochain complex

$$K^0 \xrightarrow{d_0} K^1 \xrightarrow{d_1} \ldots \to K^n \xrightarrow{d_n} 0.$$

If $\mathbf{K}^*$ is another such, $K^* + \mathbf{K}^*$ is, by definition, the complex

$$K^0 \xrightarrow{d_0} \ldots \to K^n \xrightarrow{d_n} \mathbf{K}^0 \xrightarrow{\mathbf{d}_0} \ldots \mathbf{K}^m \xrightarrow{\mathbf{d}_m} 0.$$

A complex K^* is called irreducible if it cannot be written as a sum in a nontrivial way. Now set dim $K^i = l_i$ and dim $H^i = b_i$. Here H^i denotes the ith cohomology group of K^*. Assume also that volume elements $0 \neq \omega_i \in \Lambda^{l_i}(K^i)^*$ have been chosen. Suppose K_1^*,K_2^* are two such complexes with volumes that are mutually isomorphic and that are not acyclic. Then if, in addition, K_1^*,K_2^* are irreducible, the isomorphisms $i_j{:}K_1{}^j \to K_2{}^j$ may be chosen to be volume preserving. But suppose that K_1^*,K_2^* are acyclic or that, in addition, we have chosen volume elements $0 \neq \mu_i \in \Lambda^{b_i}(H^i)^*$. If we seek volume-preserving isomorphisms that also induce volume-preserving isomorphisms in cohomology, an extra necessary condition must be satisfied: namely, for a certain invariant τ, the Reidemeister torsion, we must have $\tau(K_1^*,\omega_1,\mu_1) = \tau(K_2^*,\omega_2,\mu_2)$. If K_1^*,K_2^* are irreducible, this condition is also sufficient. $\tau(K^*,\omega,\mu)$ is defined as follows. Let $B^i = d_{i-1}(K^{i-1})$, $Z^i = \ker d_i$ and dim $B^i = t_i$. Let

$$0 \to B^i \to Z^i \xrightarrow{\pi} H^i \to 0.$$

If $\rho_i \in \Lambda^{t_i}(K^i)^*$ is any form whose restriction to B^i is non-zero, then for some $m_i \neq 0$

$$\rho_i \wedge \pi^*(\mu_i) \wedge d_i^*(\rho_{i+1}) = m_i\omega_i.$$

Define $\tau(K^*,\omega,\mu)$ by

$$\tau(K^*,\omega,\mu) = \frac{m_0}{m_1}\cdot\frac{m_2}{m_3}\cdot\ldots \qquad [1.1]$$

The definition is easily seen to be independent of the particular choices ρ_i. For many purposes it is more convenient to work with $\ln \tau$ than with τ.

Suppose $L \subset K$ is a subcomplex and $0 \to L \xrightarrow{i} K \xrightarrow{\pi} K/L \to 0$. Let ν_i,θ_i be volume elements for L^i, K^i/L^i, and suppose that volume elements for $H^*(L),H^*(K/L)$ have been chosen. If $\bar\nu_i$ is such that $i^*(\bar\nu_i) = \nu_i$ and if $\omega_i = \bar\nu_i \wedge \pi^*(\theta_i)$, then the following relation holds (see ref. 1):

$$\ln \tau(K,\omega) = \ln \tau(K,\nu) + \ln \tau(L,\theta) + \ln \tau(\mathcal{H}). \qquad [1.2]$$

Here $\mathcal{H}$ represents the long exact cohomology sequence of the pair K,L, viewed as an acyclic complex with volumes.

Now, let X be a finite C-W complex, $\mathcal{E}$ a flat orthogonal bundle over X, and $0 \neq \mu_i \in \Lambda^{b_i}[H^i(X,\mathcal{E})]^*$. Let $\omega_i \in \Lambda^{l_i}[C^i(X)]^*$ be induced by the inner product which makes the dual basis to the cells of X an orthonormal basis. Then a basic result states that $\tau(X,\omega,\mu)$ is a combinatorial invariant (see ref. 1). If, in particular, $X = M^n$ is a closed Riemannian manifold, then M^n has a combinatorial structure coming from its smooth triangulations. Moreover, $H^i(X,\mathcal{E})$ can be identified with the space of harmonic forms with coefficients in $\mathcal{E}$, and the global inner product on these forms induces a volume element for $H^i(M,\mathcal{E})$. From now on we make these choices and just write $\tau(M,\mathcal{E})$. As explained in ref. 2, by analogy with $\tau(M,\mathcal{E})$, Ray and Singer have defined a spectral invariant $T(M,\mathcal{E})$ called analytic torsion. If $\zeta_i(s) = \Sigma_{\lambda_j>0} \lambda_j{}^{-s}$ is the zeta-function of the Laplacian on i-forms with coefficients in $\mathcal{E}$, then

$$\ln T(M,\mathcal{E}) = \frac{1}{2}\sum_{i=0}^{n} (-1)^i i\zeta_i'(0). \qquad [1.3]$$

Ray and Singer conjectured that, in fact, $\tau(M,\mathcal{E}) = T(M,\mathcal{E})$. They were able to prove that $\ln \tau(M,\mathcal{E}) - \ln T(M,\mathcal{E})$ is independent of the choice of Riemannian metric; see also ref. 3 for explicit calculations in case M is a lens space. Our purpose here is to announce a proof that $\tau(M,\mathcal{E}) = T(M,\mathcal{E})$ and to describe the main ideas that are involved. Details will appear elsewhere. It has been brought to our attention that the thesis of W. Müller contains some partial results on this problem and that (private communication) he now claims to have obtained a complete solution.

THEOREM 1.1. *Let* M, $\mathcal{E}$, τ(M,$\mathcal{E}$), T(M,$\mathcal{E}$) *be as above. Then* τ(M,$\mathcal{E}$) = T(M,$\mathcal{E}$).

2. Structure of the proof

The general scheme of the argument is suggested by Hirzebruch's proof of the Signature Theorem. Note that since neither invariant depends on orientation, it suffices to prove the equality for the disjoint union $2M = \partial(M \times I)$ and $2\mathcal{E} = \mathcal{E} \times I|\partial(M \times I)$. By excising a disc from the interior of $M \times I$, we obtain a cobordism between $2M$ and the sphere S^n. Since any cobordism can be factored into a sequence of simple cobordisms, or put another way, into surgeries, it would suffice to show the following.

LEMMA 2.1. *If* M_1 *is obtained from* M_0 *by surgery on some imbedded* S^k *and if* $\mathcal{E}$ *extends over the trace of the surgery, then* $\tau(M_0,\mathcal{E}) = T(M_0,\mathcal{E})$ *implies* $\tau(M_1,\mathcal{E}) = T(M_1,\mathcal{E})$.

Lemma 2.1 would reduce the proof to checking the equality for any particular example, e.g., a flat torus for which both T and τ are easily seen to be 1. In actuality, the argument turns out to be a little more complicated than this because *Lemma 2.1* can only be proved directly in case $0 < k < n - 1$. Because these matters are neither central nor difficult, they will not be discussed any further here.

Let D^m denote the m-disc. Then there is a manifold with

boundary $\mathbf{M}$ over which $\mathcal{E}$ extends with the property that

$$M_0 = S^k \times D^{n-k} \cup \mathbf{M}$$
$$M_1 = S^{n-k-1} \times D^{k+1} \cup \mathbf{M}$$

where the union is along the boundary. Choose metrics on M_0, M_1 that induce the standard product metric (unit sphere $\times$ unit disc) on $S^k \times D^{n-k}$, $S^{n-k-1} \times D^{k+1}$. Let $N_u(S^k)$ denote the u-tubular neighborhood of S^k. Write $\mathbf{M}_u$ for $\mathbf{M}$ equipped with the metric g_u, where g_u is a family of metrics on $\mathbf{M}$ defined as follows.

$$\mathbf{M}_u = M_0 - N_u(S^k) \qquad 0 < u \leq 1/3$$
$$\mathbf{M}_u = M_1 - N_{1-u}(S^{n-k-1}) \qquad 2/3 \leq u < 1$$

For $1/3 \leq u \leq 2/3$ near $\partial\mathbf{M}_u = S^k \times S^{n-k-1}$, always make the same fixed choice of g_u independent of $\mathbf{M}$. Extend this choice to all of $\mathbf{M}$ in any fashion such that g_u defines a smooth family of metrics on $\mathbf{M}$ for $0 < u < 1$.

By considering forms ω that satisfy absolute (Neumann) boundary conditions

$$^*\omega|\partial\mathbf{M}_u = 0$$
$$^*d\omega|\partial\mathbf{M}_u = 0$$

we get a spectrum $\{\lambda_i\}$ and eigenforms $\{\phi_i\}$ for the Laplacian on $\mathbf{M}_u$. Again, the harmonic forms can be identified with $H^*(\mathbf{M},\mathcal{E})$. Thus, $\tau(\mathbf{M}_u,\mathcal{E})$, $T(\mathbf{M}_u,\mathcal{E})$ can be defined as above, but one no longer expects equality of the two invariants unless the metric is a product near the boundary $\partial\mathbf{M}$. Rather, the proper generalization of the calculation of Ray and Singer is the following. Let N be a Riemannian manifold with possibly $\partial N \neq \phi$ and write

$$e = \ln \tau(N,\mathcal{E}) - \ln T(M,\mathcal{E}).$$

THEOREM 2.1. *Let* g_u *be a one-parameter family of metrics on* N. *Then* $(d/du)e_u$ *is locally computable at the boundary. In particular,* $e_{u_2} - e_{u_1}$ *depends only on* $g_u|\partial N$.

Now let $M_0, M_1, \mathbf{M}_u$ be as above and assume $e_0 = 0$. Then

$$e_1 = (e_1 - e_{1-u}) + (e_{1-u} - e_u) + (e_u - e_0) = A_u + B_u + C_u.$$

Since we have standardized g_u near $\partial\mathbf{M}$, it follows directly from *Theorem 2.1* that B_u depends only on u and k. Thus B_u is completely independent of M_0, M_1. In essence, the remainder of the proof of *Lemma 2.1* consists of showing

PROPOSITION 2.1. *Let* $M_0, M, \mathcal{E}$ *and* $M_0', M_1', \mathcal{E}'$ *be any two triples as above. Then if* $0 \leq k < n - 1$

$$\lim_{u\to 0} (A_u - A_u') = 0.$$

If $0 < k \leq n - 1$

$$\lim_{u\to 0} (C_u - C_u') = 0.$$

Once this has been established, it follows for any M_0', M_1' that $e_1 - e_0 = e_1' - e_0'$. By choosing M_0', M_1' conveniently, $e_1' - e_0'$ can easily be shown to vanish. Then so does $e_1 - e_0$.

By symmetry, it suffices to consider C_u. Let $K = M_0$ and $L = \mathbf{M}$. Then $K/L = N(S^k)/\partial N(S^k)$. In view of **1.2**, to describe the term $\ln \tau(M_0,\mathcal{E}) - \ln \tau(\mathbf{M}_u,\mathcal{E})$ as $u \to 0$ it suffices to understand the interaction in the limit between the inner product on the harmonic forms of $\mathbf{M}_u$ and the maps of the exact sequence of the pair $M_0, \mathbf{M}_u$. To describe $\ln T(M_0,\mathcal{E}) - \ln T(\mathbf{M}_u,\mathcal{E})$ on the other hand, we must understand the limiting behavior of the non-zero spectrum of $\mathbf{M}_u$. In the next section, we give the tools that are basic for this understanding. The first two results are phrased in terms of the heat kernel.

3. Basic results

Let $E_0(x,y,t)$, $E_u(x,y,t)$ denote the heat kernels on i-forms of $M_0, \mathbf{M}_u$ with coefficients in $\mathcal{E}$. Assume $k < n - 1$.

THEOREM 3.1. *Given* $T, N, u_0 > 0$, *there exists a function* $K_N(T,u_0,u) \to 0$ *as* $u \to 0$ *such that for* $t \leq T$ *and* $x,y \in \mathbf{M}_{u_0} \subset \mathbf{M}_u$

$$\|E_0(x,y,t) - E_u(x,y,t)\| \leq K_N(T,u_0,u) \cdot t^N.$$

Now let A^{n-k}_{u,u_0} denote the annulus $S^{n-k-1} \times I$ with radii u,u_0, where $u < u_0$. Let $E_{u,u_0}(x,y,t)$ denote the heat kernel on $A_{u,u_0} \times S^k$ with absolute boundary conditions at both boundary components.

THEOREM 3.2. *Given* T,N, *there exists* $C_N(T)$ *and* $K(T,u_0,u)$ *as above such that*

$$\left|\int_{A_{u,u_0}\times S^k} [E_{u,u_0}(x,x,t) - E_u(x,x,t)]\right| \leq [K_N(T,u_0,u) + C_N(T)u_0^{n-k}]t^N.$$

Both of the results just stated are rather easy consequences of a sharp version of the Sobolev inequality on $\mathbf{M}_u$. This is the basic estimate on which everything else depends. Before stating it, we introduce some terminology. An $(i + j)$ form w on $S^k \times D^{n-k}$ is said to be of type (i,j) if at each point $(x_1,x_2) \in S^k \times D^{n-k}$ there is an i-form $\phi_1(x_1,x_2)$ tangent to $S^k \times x_2$ and a j-form tangent to $x_1 \times D^{n-k}$ such that $w = \phi_1 \wedge \phi_2$. Then for any ω_l, we can write $\omega_l = \sum_{j\leq i\leq l} \omega_{i,l-i}$ with $\omega_{i,l-i}$ of type $(i,l-i)$. Let $\|x\| = \rho(x,\partial\mathbf{M}_u) + u$. Here ρ denotes distance. Let $d\theta$ denote the solid angle form on $A_{u,1}$, normalized so that $\int_{S^{n-k-1}} d\theta = 1$. Note that $d\theta$ is harmonic and satisfies absolute boundary conditions. Set

$$\|d\theta\|_u^2 = \int_{A_{u,1}} d\theta \wedge {}^*d\theta$$

Then

$$\frac{\|d\theta(x)\|}{\|d\theta\|_u} = \sqrt{V}(S^{n-k-1}) \cdot \|x\|^{1-n+k} \cdot \begin{cases} \dfrac{(2+k-n)^{1/2}}{(u^{2+k-n}-1)^{1/2}} & n-k>2 \\ |\log u|^{-1/2} & n-k=2 \end{cases}. \qquad [3.1]$$

According to **3.1**, we cannot always expect the constant in the Sobolev inequality to stay bounded independent of u. As it turns out, the crucial point is that the constant blows up at a rate slower than that at which the area of the inner boundary of $A_{u,1}$ goes to zero.

THEOREM 3.3. *There exists* K *such that for* $x \in \mathbf{M}_u$, $i \neq n-k-1$ *and all* u

$$(1)\ \|\omega_{i,l-i}(x)\| \leq K \cdot \sum_{j=0}^{N>n/2} \|\Delta^j\omega\|_{\mathbf{M}_u}$$

$$(2)\ \|\omega_{l-n+k+1,n-k-1}(x)\| \leq K\left(1 + \frac{\|d\theta(x)\|}{\|d\theta\|_u}\right) \sum_{j=0}^{N>n/2} \|\Delta^j\omega\|_{\mathbf{M}_u}.$$

4. Consequences of the basic results

We will not explain in detail how *Theorems 3.1, 3.2,* and *3.3* lead to a proof of *Proposition 2.1*. We will, however, list all the

results on which the proof depends. To simplify the statements and without real loss of generality, we assume real coefficients, i.e., $\mathscr{E}$ is the trivial flat line bundle. First recall the relation between the cohomology of M_0 and that of $\mathbf{M}$. The pair $(M_0,\mathbf{M})$ is equivalent by excision to the pair $[N(S^k),\partial N(S^k)]$.

$$H^i[N(S^k),\partial N(S^k),R] = \begin{cases} 0 & i \neq n-k,n \\ R & i = n-k,n \end{cases}$$

Then the exact sequence of the pair gives

$$0 \to H^i(M_0,R) \xrightarrow{l} H(\mathbf{M}_u,R) \to 0 \qquad i \neq n-k-1, n-k, n \quad [4.1]$$

$$0 \to R \xrightarrow{j} H^n(M_0,R) \xrightarrow{l} H^n(\mathbf{M}_u,R) \to 0 \quad [4.2]$$

$$0 \xrightarrow{j} H^{n-k-1}(M_0,R) \xrightarrow{l} H^{n-k-1}(\mathbf{M}_u,R) \xrightarrow{\delta} R \xrightarrow{j} H^{n-k}(M_0,R) \xrightarrow{l} H^{n-k}(\mathbf{M}_u,R) \to 0. \quad [4.3]$$

R has its usual norm, and all cohomology groups are equipped with the inner product coming from the identification with harmonic forms. There are two cases to consider

(*a*) $j{:}R \to H^{n-k}(M_0,R)$ is the zero map.
(*b*) j is an isomorphism.

PROPOSITION 4.1.
(1) *If* $i \neq n-k-1, n-k, n-1, n$, *then there exists* $K,\lambda > 0$ *such that for* $t \geq 1$ *and all* n

$$\mathrm{tr}[E_u(t)] \leq \dim H(\mathbf{M}_u,R) + Ke^{-\lambda t}.$$

In particular, the smallest non-zero eigenvalue stays bounded away from zero as $u \to 0$.
(2) *If* $i = n$, *let* λ_u *denote the smallest non-zero eigenvalue and let* ϕ_u *be the corresponding eigenform. Then* ϕ_u *is exact and* $\delta\phi_u$ *is an eigenform in dimension* $n-1$ *with eigenvalue* λ_u. *As* $u \to 0$ *then* $\lambda_u \to 0$.
(3) *There exist* $K,\lambda > 0$ *such that, if* $i = n-1$, n, *then for* $t \geq 1$ *and all* u

$$\mathrm{tr}[E_u(t)] \leq \dim H^i(\mathbf{M}_u,R) + e^{-\lambda_u t} + K \cdot e^{-\lambda t}.$$

In particular, the next eigenvalue after λ_u *stays bounded away from zero as* $u \to 0$.
(4) *In case* (a), (1) *holds for* $i = n-k-1, n-k$ *as well.*
(5) *In case* (b), (2) *holds with* $n-1$, n *replaced by* $n-k-1, n-k$.

PROPOSITION 4.2. *As* $u \to 0$, *in the limit:*
(1) *If* $i \neq n-k-1, n-k$, n, *then* l *becomes an isometry.*
(2) *If* $i = n$, *then* $l|j(R)^{\perp}$ *becomes an isometry.*
(3) *In case* (a), *if* $i = n-k$, *then* l *becomes an isometry. If* $i = n-k-1$, *then* l *becomes an isometric injection.*
Let ω_i *be the harmonic* i*-form on* S^k *having period* 1, $i = 0,k$. *If* $h_{n-k-1} \in l[H^{n-k-1}(M_0,R)]^{\perp} \subset H^{n-k-1}(\mathbf{M}_u,R)$, *then*

$$\lim_{u\to 0} \frac{\|\delta(h_{n-k-1})\|^2}{\|h_{n-k-1}\|_u^2} \cdot \|d\theta\|_u^2 \|\omega_0\|^2 = 1.$$

(4) *In case* (b), *if* $i = n-k-1$, *the* j *becomes an isometry. If* $i = n-k$, *then* $l|j(R)^{\perp} \subset H^{n-k}(M_0,R)$ *becomes an isometry.*

Part (3) *of Proposition 4.2* has the consequence that $(\|h_{n-k-1}(x_0)\|/\|h_{n-k-1}\|_u) \to 0$ as $u \to 0$ for any fixed $x_0 \in \mathrm{int}(\mathbf{M}_u)$. Note that this is consistent with the implications of *Theorem 3.1* for large t. For small u, $h_{n-k-1}/\|h_{n-k-1}\|$ *and* $\delta\phi_u/\|\delta\phi_u\|$ behave like $d\theta/\|d\theta\|_u$.

PROPOSITION 4.3.
(1) *Let* λ_u *be as in* (2) *of* Proposition 4.1. *Let* ω_i *be the harmonic form on* S^k *with period* 1, $i = 0,k$. $H^n(M_0,R) \ni h_n = j(1)$. *Then*

$$\lim_{u\to 0} \lambda_u \Big/ \frac{\|h_n\|^2}{\|d\theta\|_u^2 \cdot \|\omega_0\|^2} = 1.$$

(2) *In case* (b), *let* λ_u, *be as in* (5) *of* Proposition 4.1. *Let* $H^{n-k}(M_0,R) \ni h_{n-k} = j(1)$. *Then*

$$\lim_{u\to 0} \lambda_u \Big/ \frac{\|h_{n-k}\|^2}{\|d\theta\|_u^2 \cdot \|\omega_0\|^2} = 1.$$

Theorems 3.1 and *3.2* may be brought to bear on *Propositions 4.1–4.3* by considering t so large that $E_0(x,y,t)$ is almost a projection on the harmonic subspace. The proofs then become mostly formal consequences of *Theorems 3.1, 3.2,* and *3.3* together with some simple geometric arguments.

Proposition 2.1 is a quite straightforward consequence of the results of Sections 3 and 4. Here, we mention only one point of particular interest. In order to compute

$$\lim_{u\to 0} [\ln \tau(M_0) - \ln \tau(\mathbf{M}_u)]$$

we apply **1.2** to the pair $M_0\mathbf{M}_u$. In view of *Proposition 4.2,* in case (*a*) only the following terms are of interest in computing $\ln \tau(\mathscr{H})$

$$\lim_{u\to 0} [\ln\|\delta(h_{n-k-1})\| - \ln\|h_{n-k-1}\|_u] \quad [4.4]$$

$$\ln\|h_n\| \quad [4.5]$$

By (*3*) of *Proposition 4.2,* **4.4,** although not finite, is independent of M_0,M_1; **4.5,** on the other hand, clearly is not. But in order to compute

$$\lim_{u\to 0} [\ln T(M_0) - \ln T(\mathbf{M}_u)]$$

we must consider also the term

$$\lim_{u\to 0} \tfrac{1}{2} \ln \lambda_u$$

which, by (*1*) of *Proposition 4.3,* is just

$$\ln\|h_n\| - \ln\|d\theta\|_u - \ln\|\omega_k\|. \quad [4.6]$$

In computing $(e_0 - e_u)$, the nonlocal terms $\ln\|h_n\|$ in **4.5** and **4.6** cancel one another, leaving a contribution that is purely local. In case (*b*) something similar also occurs in dimension $n-k$.

5. Manifolds with boundary

If we consider forms satisfying absolute boundary conditions, then τ and T make sense, but as we have stated previously, they are not equal in general. It is possible to extend the techniques described here so as to cover this case. The argument involves replacing M by its double and doing everything equivariantly with respect to the natural involution

THEOREM 5.1. *There is an invariant* $C(\partial M,\mathscr{E})$ *that is lo-*

cally computable at the boundary and that vanishes if up to sufficiently higher order the metric is a product at the boundary, such that

$$\ln T(M,\mathcal{E}) = \ln \tau(M,\mathcal{E}) + C(\partial M,\mathcal{E}).$$

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