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Social dynamics and the quantifying of social forces

(logistic curve/replacement and evolution/transportation)

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ABSTRACT **Social and industrial evolutionary processes are considered to be a sequence of replacements or substitutions: new ideas for old, new labor patterns for old, new technologies for old. The logistic equation has often been used to describe population growth processes and replacement processes. It sometimes suffers from contradicting observational data. It is shown here that the deviations are often associated with unusual intermittent events—wars, strikes, economic panics, etc.—and that in many cases a few years after the event it can be abstracted as an instantaneous δ function impulse. After the event, the evolutionary process continues along its normal course. A formula is derived to use the observational data to determine the strength of the impulse modeling an event.**

During any period of history, the middle-aged and elderly lament upon the changes "in the world" during their lifetime, complaining that "things" are no longer what they were. The things that have changed the most vary from one generation to another. In this century the automobile has replaced the horse; the light bulb, the gas mantle; the movie theatre, home entertainment (finally, television, the movie theatre); the Balkan States of the Soviet Bloc, the old Austro-Hungarian Empire; the meaningful relationship, formal marriage; to name just a few. The continuing change in our habits and loyalties is bound to affect economic and social structures. On this basis, social science models should include the evolution of social processes.

The aim of this paper is to present an elementary point of view for the development of a style for the description of social dynamics. A physical scientist will immediately observe the influence of Newton's laws of particle dynamics upon the structure of this style. Following in the Newtonian tradition, I propose three laws of social dynamics (1).

THREE "LAWS" OF SOCIAL DYNAMICS

Newton's first law of dynamics is the postulate that, in the absence of an external force, every body in a state of motion will remain in that state of motion—i.e., it will continue to move in a straight line with a constant velocity. Of course, this situation never prevails in earthbound experiments, but it is still a good starting point for the construction of mathematical models of dynamical systems.

The first law of social dynamics is here chosen to have two similarly stated parts

(*a*) In the absence of any social, economic, or ecological force, the rate of change of the logarithm of a population, $N(t)$, of an "organism" is constant,

$$d\log N(t)/dt = \text{constant}. \qquad [\mathbf{1a}]$$

Without the prescribed forces this equation is also postulated to be valid for the variation of the population of objects of production (automobiles, radios, etc.), and for the change in a population of a new social group.

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(*b*) In the absence of any social, economic, or ecological force, the rate of change of the logarithm of the price of maintenance $P(t)$ (per unit time) of an "organism" is also constant

$$d\log P(t)/dt = \text{constant}. \qquad [\mathbf{1b}]$$

In the case of objects of production, $P(t)$ is to be interpreted as a unit cost.

A discussion of part *b* of the first law and of the influence of prices on part *a* will be given elsewhere.

The population of inanimate objects is included so that population growth models might be applied to production of and competition between manufactured items.

Eq. **1a** is, of course, nothing but the Malthusian law of exponentiation of populations (2), and Eq. **1b** is a statement of the accountants' "discounting" principle and the housewives' observation that things are always getting more expensive. The constant in Eq. **1a** might be negative as well as positive because interest in some items just dies away.

It might be claimed that the first law of social dynamics is more often applicable to the real world than is Newton's first law of mechanics to real dynamical systems, since social reformers as well as conservative politicians frequently seek means of violating Eqs. **1a** and **1b**. Numerous observations exhibiting the first law are in refs. 3 and 4.

Newton was very astute in his use of the second law as the definition of a force. Who can go wrong by making definitions if he does not make too many of them? The second law is just the statement that a force is that which causes the first law to be violated. I will not attempt to outdo the master on this point.

The second "law" of social dynamics is the postulate:

Eq. **1a** or **1b** or both are violated when a social, economic, or ecological force is applied. How is the force to be chosen or measured? By observing the manner in which the first law is violated!

Several examples of applied forces will follow the statement of:

The third "law." Evolution is the result of a sequence of replacements.

Newton never tried to derive his laws of force from first principles. The restoring force for a displaced mass in a harmonic oscillator was merely the force of greatest mathematical simplicity and one found to be useful in describing many physical phenomena. The postulation of the inverse square law of the gravitation force was natural for a genius whose geometric intuition told him it was just what was required to produce Kepler's empirical observations on elliptical planetary orbits.

One of the simplest mathematical forms for a force that might replace the constant on the right-hand side of Eq. **1a** is a linear force:

$$F\{N(t)\} \equiv k - \alpha N(t) \qquad [\mathbf{2}]$$

which represents a deterence to population growth. If one sets

$\alpha = k/\theta$, then Eq. **1a** becomes the well-known Verhulst equation (with population saturation at $N = \theta$) (3, 5)

$$d\log\{N(t)/\theta\}/dt = k\{1 - (N(t)/\theta)\} \qquad [3]$$

whose solution yields a remarkably accurate representation of the population growth in many countries. The growth to saturation according to the solution of Eq. 3 is often called the logistic curve. The characterization of deviations will be discussed in the next section.

If one sets $x \equiv N/\theta$ and $y = 1/x$, then

$$-d(y-1)/dt = k(y-1), \text{ and} \qquad [4]$$

$$y(t) - 1 = [y(0) - 1]\exp(-kt) \qquad [5]$$

so that

$$\log[x/(1-x)] = \log[x(0)/[1-x(0)]] + kt. \qquad [6]$$

A more general (two-parameter) force law (3) is

$$F\{N(t)\} = k\{1 - [N(t)/\theta]^{\nu}\}/\nu \equiv G_{\nu}(N/\theta) \qquad [7]$$

which yields Verhulst's equation when $\nu = 1$ and the Gompertz equation (6) when $\nu = 0$:

$$dN/dt = -Nk\log(N/\theta). \qquad [8]$$

The simplest interaction between two species is obtained by choosing the second species to apply a force linear in its population to the first species and vice versa so that

$$d\log N_1/dt = k_1 + C_{12}N_2$$

$$d\log N_2/dt = k_2 + C_{21}N_1.$$

The case $k_1 = \alpha_1 > 0$, $k_2 = \alpha_2 < 0$, $C_{12} = \lambda_1 < 0$, and $C_{21} = \lambda_2 > 0$ yields the well-known Lotka–Volterra equations for competing species (7–9)

$$dN_1/dt = \alpha_1 N_1 - \lambda_1 N_1 N_2 \qquad [9a]$$

$$dN_2/dt = -\alpha_2 N_2 + \lambda_2 N_1 N_2 \qquad [9b]$$

such that the predator species 2 preys upon species 1 in such a manner that species 2 would disappear without species 1 and species 1 would propagate in a Malthusian manner without species 2. These equations have periodic solutions (8, 9).

Finally, in an assembly of many interacting species, one might add the influence of other species to the right-hand side of Eq. 3 as a random force, $F(t)$, to yield the equation

$$d\log[N(t)/\theta]/dt = kG_{\nu}(N/\theta) + F(t). \qquad [10]$$

The special case $G_1(x) = 1 - x$ was first considered by Leigh (10) and the general case was analyzed in ref. 9. If one assumes $F(t)$ to be generated by a Gaussian random process with

$$\langle F(t_1)F(t_2)\rangle_{av} = \sigma^2\delta(t_2 - t_1), \qquad [11]$$

then it can be shown in the standard manner that the probability distribution of N at time t, $P[N(t)]$, satisfies the Fokker–Planck equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial v}k\{PG(\exp v)\} + \frac{1}{2}\frac{\partial^2 P}{\partial v^2}\sigma^2 \qquad [12a]$$

in which

$$v = \log(N/\theta). \qquad [12b]$$

The steady-state solution of this equation with $\dot{P} = 0$ is easily verified (9) to be

$$P(v,t) = P_0\exp\left[2\sigma^{-2}\int_0^v G(\exp v)dv\right] \qquad [13]$$

in which P_0 is a normalization constant. In the Gompertz and Verhulst cases, the steady-state distribution functions are, respectively,

$$\text{Gompertz: } P = P_0\exp(-kv^2/\sigma^2),\ v = \log(N/\theta) \qquad [14]$$

$$\text{Verhulst: } P = P_0\exp 2k[v - \exp v]/\sigma^2$$
$$= P_0(N/\theta)^{2k/\sigma^2}\exp(-2kN/\theta\sigma^2). \qquad [15]$$

Eq. **15** was first derived by Leigh and Eq. **14** was first given in ref. 9. A noisy Gompertz growth process can be shown to be equivalent to the Ornstein–Uhlenbeck process for the theory of Brownian motion in the variable v (11).

Of course, many more models have been discussed in the literature. My main interest here is to discuss the third law, the law of evolution, as a sequence of replacements. I choose the canonical form for the evolutionary force to be the right-hand side of the Verhulst (or logistic) equation (Eq. 3). This was exploited by Fisher and Pry (12) as the appropriate model for the dynamics of industrial replacement. I discuss the replacement as an evolutionary process in the next section where I shall also analyze cases in which the canonical form is violated.

DYNAMICS OF REPLACEMENT PROCESS AND MEASUREMENT OF FORCES ACCELERATING AND DETERRING REPLACEMENT OR EVOLUTIONARY PROCESS

A set of Fisher and Pry's typical evolutionary curves is plotted in Fig. 1. They represent the replacement of one industrial process (or product) by another. They can also be interpreted as market penetration curves. Here x is the fraction of the market captured by the new process (or product) and $(1 - x)$ is that remaining for the old. Fisher and Pry (12) have found over twenty examples of their industrial replacement mechanism and other authors have supplemented the list (13–15).

Herman and I (16) have noticed that the industrial revolution in both the United States and Sweden has also evolved in the logistic pattern, as exhibited in Fig. 2, with $(1 - x)$ there being the fraction of the work force performing agricultural tasks and x being the fraction performing nonagricultural tasks. The data

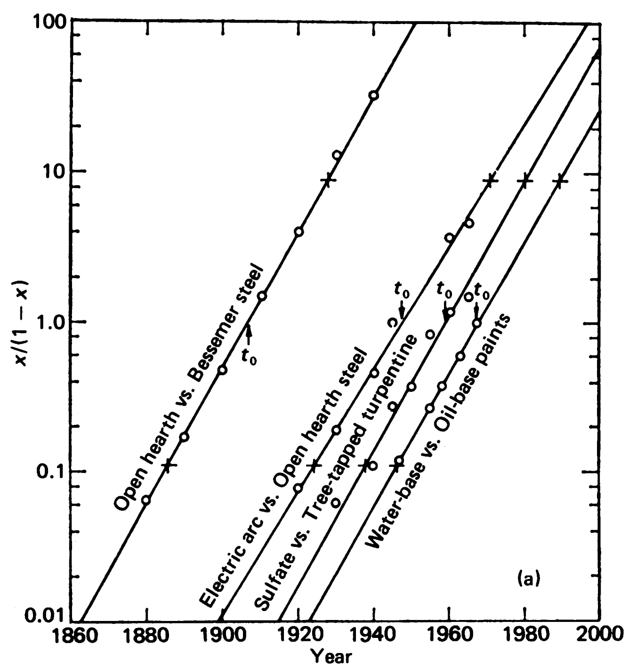

FIG. 1. Replacement dynamics of four technologies, following the model and data of Fisher and Pry (12).

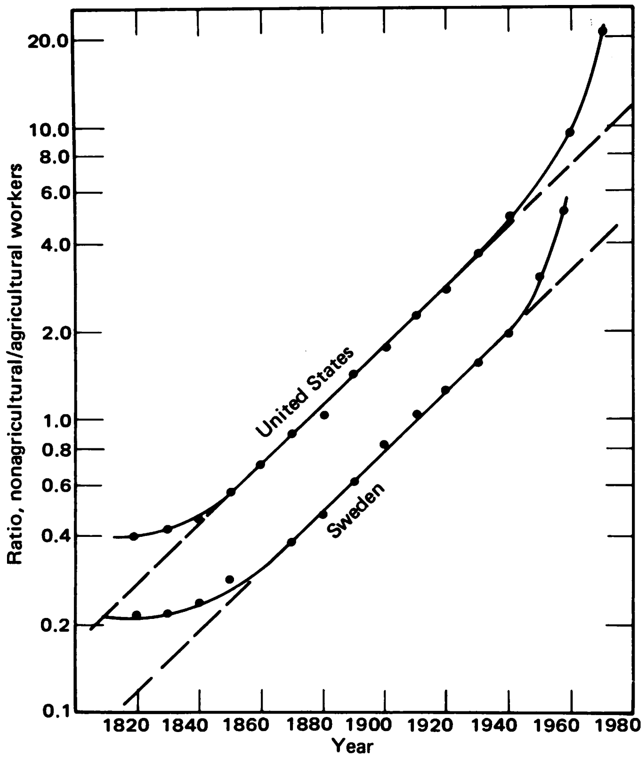

FIG. 2. Variation of the ratio of nonagricultural workers to agricultural workers in the American and Swedish labor force (16).

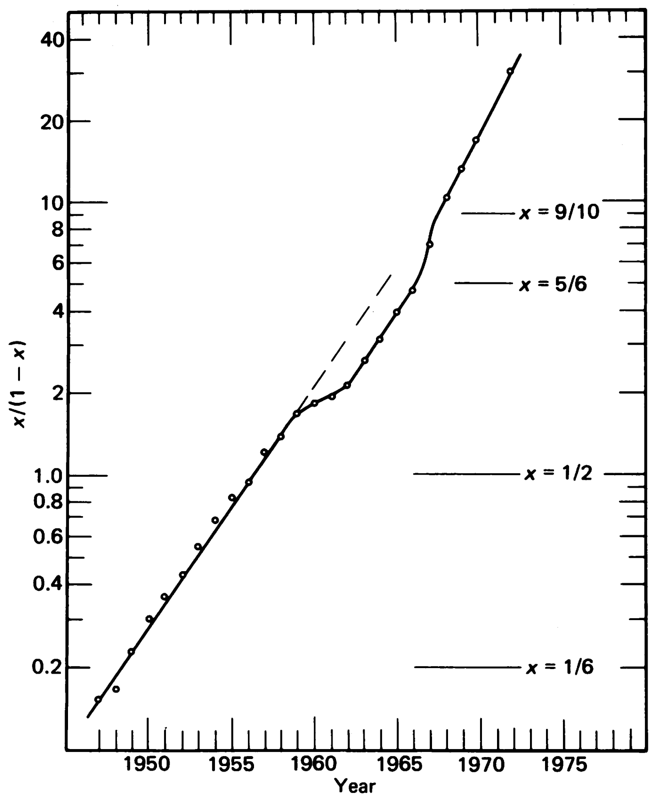

FIG. 3. Manner in which intercity passenger travel by rail has been replaced by air travel.†

follow the solution of the Verhulst equation for over 100 years. A new social force emerged in 1940 through a flood of telegrams from the White House, starting with the word "Greetings . . ." and received by young farmers the country over. These telegrams accelerated the decline in agricultural work force beyond the expected rate of the canonical Verhulst process.

An example of the influence of a deterring force is evident in Fig. 3, a record of the replacement of rail service by air service in intercity passenger travel.† There,

$$x = \frac{\text{annual air passenger miles}}{\text{annual air \& rail passenger miles.}} \qquad [16]$$

Notice that in the period 1947–1959 the data are in accord with Eq. **6b.** The deterrence in replacement evolution during 1960–1961 seems to have been due to several unusually long strikes by airline workers and to the public's response to a series of serious airplane accidents. By 1962, the system recovered and the replacement curve continued with its old slope until the late 1960s. Then, accelerated replacement to the end of the decade occurred at the time the largest rail passenger carrier, the Penn-Central line, was suffering through its prebankruptcy and bankruptcy condition. The management of passenger rail service was reorganized in the 1970s by Amtrak.

The role of intercity buses was omitted from the above discussion because, during the period investigated, the fraction of the total passenger traffic maintained by them changed very little.

Other examples of intermittent accelerating and deterring forces are easily found. Deviations from the logistic national population curves during and after wars and during economic depressions are common. These will be discussed elsewhere. R. Bolt (personal communication) has shown that the United States has been experiencing an educational evolution since 1900 in that the fraction of persons receiving university degrees has been following a logistic pattern with intermittent forces appearing during and after World War I and World War II. The replacement of the sailboat by the steamboat is discussed at the end of this section. I now present a systematic procedure for measuring the magnitude of intermittent forces.

Let the solid straight line in Fig. 4 represent the canonical evolutionary curve for a replacement process and suppose that an accelerating force was applied at time τ and withdrawn at a later time. After withdrawal, the evolutionary curve will parallel the unaccelerated one and will reach a level A at a time t^*, somewhat earlier than the time t that would have been required to reach the same level in the normal fashion. If $F(t)$ is the accelerating force, the accelerating process is characterized by

$$d\log x/dt = k(1 - x) + F(t). \qquad [17]$$

In the time regime after the withdrawal of an intermittent force, the continuing evolutionary curve parallels the unaccelerated one. The time gained,

$$\Delta t = t - t^* \qquad [18]$$

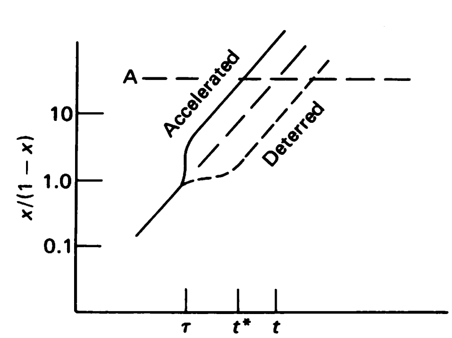

FIG. 4. Schematic of the manner in which an intermittent accelerating or deterring force affects replacement dynamics.

† Montroll, E. W. (1978) *Proceedings of Environmental Protection Agency Conference on Mathematical Modeling*, in press.

in reaching the level A could have been achieved by the introduction of an equivalent δ function impulse. The object of the remainder of this section is to derive an expression for the amplitude α of the δ function impulse

$$F(t) = \alpha\delta(t - \tau) \qquad [19]$$

necessary to yield the observed gained time Δt. We then consider α to be the measure of the magnitude of the originally applied accelerating force, whatever its detailed nature. The form used for the δ function is

$$F(t) = \begin{cases} 0 \text{ if } t < \tau \text{ or } \tau + \delta < t \\ \alpha/\delta \text{ if } \tau \le t \le \tau + \delta \end{cases}. \qquad [20]$$

Our final required equations will be obtained by taking the limit $\delta \to 0$. The solution of a canonical Verhulst equation (Eq. 3) was shown to be Eqs. **5** and **6**.

Upon introduction of the δ function force, the form of Eq. **17** is

$$dx/dt = kx(1 - x) + x\alpha/\delta \quad \text{with } \tau < t < \tau + \delta \equiv T, \text{ or} \qquad [21a]$$

$$-dy/dt = y\{k + (\alpha/\delta)\} - k = k(y - 1) + (y\alpha/\delta), \quad \text{if } y = 1/x. \qquad [21b]$$

In the regime $t > T$, Eq. 3 is again applicable with its solution being

$$\log\{x(t^*)/[1 - x(t^*)]\} = -\log[y(T) - 1] + k(t^* - T) \qquad [22]$$

at the time t^* when the accelerated evolution reaches level A in Fig. 4. Then from Eqs. **22** and **6**, because the function on the left side of both equations corresponds to the level A,

$$k\Delta t \equiv k(t - t^*) = -\log\{[y(T) - 1]/[y(0) - 1]\} - kT. \qquad [23]$$

The quantity $y(T)$ is found by solving the accelerated evolution equation (Eq. **21b**). Before doing so, we note that

$$\log\left\{\frac{y(T)-1}{y(0)-1}\right\} = \log\left\{\frac{y(\tau)-1}{y(0)-1}\right\}\cdot\left\{\frac{y(T)-1}{y(\tau)-1}\right\}$$

$$= -k\tau + \log\{[y(T) - 1]/(y(\tau) - 1)\} \qquad [24]$$

$$k(t - t^*) = -\log\left\{\frac{y(T)-1}{y(\tau)-1}\right\} - k\delta,$$

since

$$T - \tau = \delta. \qquad [25]$$

Since δ is small

$$y(T) - 1 = y(\tau) - 1 + \delta dy/dt + \ldots.$$

By using Eq. **21b**, as $\delta \to 0$

$$\log\left\{\frac{y(T)-1}{y(\tau)-1}\right\} = \log\left\{1 + \frac{\delta\, dy/dt}{y(\tau)-1}\right\} \sim \delta(dy/dt)/[y(\tau) - 1]$$

$$\to \alpha y(\tau)/[1 - y(\tau)]. \qquad [26]$$

Then, since $y(\tau) = 1/x(\tau)$, we have, from Eq. **25**, as $\delta \to 0$

$$k\Delta t \equiv k(t - t^*) = \alpha/[1 - x(\tau)] \qquad [27a]$$

or

$$\alpha = k(\Delta t)[1 - x(\tau)]. \qquad [27b]$$

Incidentally, the general solution of Eq. **17** is

$$x(t) = x(0)\left\{\exp\int_0^t [k + F(t)]dt\right\} \times \left\{1 + kx(0)\int_0^t d\tau \exp\int_0^\tau [k + F(t')]dt'\right\}^{-1}$$

Eq. **27b** can be derived by incorporating Eq. **19** in this expression. However, the algebra is more tedious than that given above.

Let us calculate the value of α appropriate for the deterring intermittent force applied in the period 1960–1961 to airline evolution as indicated in Fig. 3. The Δt value at the level $x = 5/6$ is -1.8 years; $x/(1 - x) = 1.7$ or $x = 0.63$ at the time of the application of the force. To estimate k, note that 11 years were required for $x/(1 - x)$ to be multiplied by a factor 10 in evolving from 0.1 to 1.0 so that, from Eq. **6b**,

$$k = (1/11 \text{ year}) \log_e 10 = 0.209/\text{year}.$$

Hence, from Eq. **27b**

$$\alpha = -(0.209)(1.8)(0.37) = -0.139 \simeq -0.14.$$

The replacement pattern of sail by steam in the U.S. Merchant Marine during the period 1830–1950 was rather like that of rail by air travel in the middle 1900s. Data from tables Q417–432 of ref. 17 are summarized in Fig. 5. The launching of steam ships in significant numbers dates from 1825. During the two decades 1830–1850, the logarithm of the fraction of tonnage in steamers to that in sailing ships followed the canonical straight line, as it did again in the interval 1870–1915. At first, steamboats appeared in river traffic and then in coastal traffic; later on they began transatlantic runs, but only after the Civil War could they successfully compete with the clipper ships on the longer Pacific passages and on the voyages around the Horn connecting the East Coast with San Francisco.

The first fast clipper ships appeared in the 1830s, at about the same time steamboats were becoming practical. They were

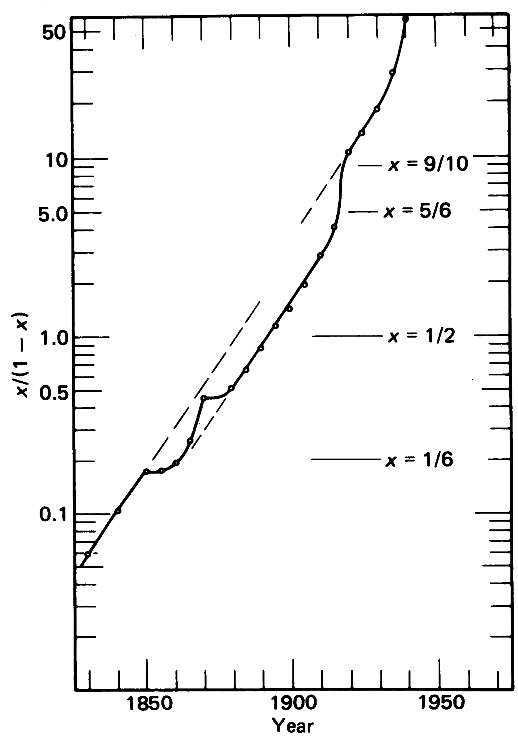

FIG. 5. Manner of replacement of sailing ships by steamships in the U.S. merchant fleet; data from ref. 17.

Table 1. Gross tonnage, according to type, in United States merchant fleet*

Year	Gross tonnage, tons × 10^{-3} Steam	Sail
1840	202	1978
1845	326	2091
1850	526	3010
1855	770	4442
1860	868	4480

* Data from ref. 17.

built in large numbers during the decade 1845–55 (peaking between 1850 and 1853). Two important events of 1849 stimulated their production, perturbing the take-over by steam: (*i*) discovery of gold in California; and (*ii*) repeal of the British Navigation Acts and the breaking of the China trade monopoly long enjoyed by the British merchant marine. The expansion of trade to the West Coast, the Orient, and Australia by adventurous American skippers created an enormous demand for the speedy clippers as indicated in Table 1.

"Among the most famous builders of Clipper ships was Donald McKay of Boston who launched the Sovereign of the Sea of 2421 tons registered in 1852. . . . this Clipper achieved a speed of 411 miles in one day. But it was [his] Lightening (1854) . . . that established the best record for a single days run, 436 miles, a record not surpassed by a steam ship for many years . . . . Construction of the extreme Clipper capable of making up to 18 knots generally stopped after 1854 because of a financial slump" (18). During the tight money episode of the Panic of 1857, all types of construction were curtailed.

Steamboat construction was favored over sail for the increased local transport required by the Civil War. "In order to keep up with the times [even] Donald McKay, during the Civil War, changed his yard over so that he could build iron ships, marine engines, etc." (19). The considerably improved steamboat models dominated naval construction in the postwar reconstruction period 1865–1873 only to be abated by the Panic of 1873 whose effects persisted for several years. With the return to normalcy, the steamboat replacement curve followed its canonical course until 1915 when the shipping requirements of World War I stimulated an accelerated naval construction program. By that time, no one considered new sailing vessels as suitable for commercial shipping.

Now, let us estimate the magnitude of an instantaneous impulsive force that would, in the long run, have the deterrent effect observed from the combination of two panics (1857 and 1873), the Civil War, and Mr. McKay's bold attempt to demonstrate the superiority of sail to steam. The deterrence time of these combined influences, measured at $x = \frac{1}{2}$ is $t - t^* = -11.25$ years. The value of k appropriate for the rate constant in a Verhulst equation to reproduce observed $x/(1-x)$ values in 1830 and 1850 is $k = 0.052$/year. The $x(t)$ to be identified in Fig. 5 with 1850 (the year of the application of the deterring force) is 0.145. Hence, from Eq. **27b**,

$$\alpha = (0.052)(-11.25)[1 - 0.145] = -0.50$$

A number of α values for other processes will be presented elsewhere.

NATURE OF CONTINUING PROGRAM

A further elaboration of the above ideas is possible through the following data analysis program. The values of the rate constants, k, appropriate to numerous industrial (12–15), scientific (20), and social (16–18) evolutionary processes as well as those of population growth (3, 21) can be assembled. α values can be derived from those evolutionary curves that indicate the existence of intermittent forces. It is hoped that, as the collection of k and α values grows, one will develop some intuition on the evolutionary rate constants associated with various classes of processes and the influence of various types of accelerating and deterring forces. In cases exhibiting a significant random component to a force law (cf. Eq. **10**) the statistical parameters can be estimated.

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