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The dimension of product spaces

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ABSTRACT The covering dimension of a product space can exceed the sum of the dimensions of its factors. A separable metric space X and a paracompact space Y are constructed such that $\dim X + \dim Y = 0 < 1 = \dim(X \times Y)$. Related results are also discussed.

The classical inequality

$$\dim(X \times Y) \leq \dim X + \dim Y \qquad [1]$$

has been intensively studied and has been shown to hold for large classes of spaces. In particular, it holds when X and Y are metric (1), when X is compact (2), and when X is locally compact and paracompact (3). Other positive results along with a discussion of the problem can be found in refs. 4–9. Strict inequality in [1] was shown possible by Pontrjagin's 1930 example (10) of a pair of two-dimensional compact metric spaces whose product has dimension three. In this note I show that inequality 1 fails to hold in general, even when X is separable metric and Y is paracompact.

All spaces are assumed to be completely regular. By dimension I mean the Lebesgue covering dimension as modified by Čech—i.e., $\dim X$ is the least integer n such that every finite cozero cover of X can be refined to a finite cozero cover of order $n + 1$. (A set is called *cozero* if it can be represented in the form $\{x | f(x) \neq 0\}$ for some continuous function f. A cover has *order* $n + 1$ if no point is contained in more than $n + 1$ elements of the cover.) Lebesgue's original definition of dimension (which can be obtained by replacing "cozero" above by "open") agrees with the Čech–Lebesgue definition in normal spaces but is inappropriate for spaces that are not normal. Some dimension theorists still use Lebesgue's original definition and restrict their attention to normal spaces. Since X, Y and $X \times Y$ are all normal in example 2, we have a counterexample even in this restricted case.

Example 1: There exists a separable metric space X and a paracompact space Y such that $\dim X + \dim Y = 0 < 1 = \dim(X \times Y)$.

Construction. We will use C to denote the Cantor set and $\mathcal{E}$ to denote its usual topology. Because $(C,\mathcal{E})$ is homeomorphic to $\prod_{q \epsilon Q}\{0,1\}_q$, it can be represented in a natural way by the power set of the rationals, $\mathcal{P}(Q)$, by identifying each function in $\prod_{q \epsilon Q}\{0,1\}_q$ with its support. Under this representation, the supremum function, $\sup c$, makes sense for each $c \epsilon C$ and maps C onto the extended real line, $[-\infty,\infty]$. Let ρ be the topology on C generated by sup and $\mathcal{E}$—i.e., generated by the subbase $\mathcal{E} \cup \{\sup^{-1}U\colon U \text{ is open in } [-\infty,\infty]\}$. It can be shown that ρ is a complete separable metric topology.

A subset of a topological space is called a *Bernstein set* if it intersects each uncountable closed set yet does not contain any uncountable closed set. Every complete metric space contains a Bernstein set (see ref. 11). Let A be a Bernstein subset of (C,ρ). All we need to construct example 1 is a suitable locally countable refinement of the restriction of ρ to A, $\rho|_A$, which we will call ρ^*. Any of the standard inductive refinements (for example Kunen's in ref. 12) would work, but for the readers not familiar with these constructions we will use the following (less elegant but simpler) definition of ρ^*. Fix a countable ρ-dense subset, D, of A. Assign the points of $A\backslash D$ to subsets of D (sending a to D_a) in such a way that

(*i*) $a \,\epsilon\, c\ell_\rho D_a \cap c\ell_\rho(D\backslash D_a)$ for each $a \,\epsilon\, A\backslash D$, and

(*ii*) if $E \subset D$ and $|(A\backslash D) \cap c\ell_\rho E \cap c\ell_\rho(D\backslash E)| = 2^\omega$ then $E = D_a$ for some $a \,\epsilon\, A\backslash D$.

(To do this, simply well order the collection of all $E \subset D$ that satisfy (*ii*) above as $\{E_\alpha\colon \alpha < 2^\omega\}$ and, for each $\alpha < 2^\omega$, recursively choose $a_\alpha \,\epsilon\, (A\backslash D) \cap \mathcal{C}\ell_\rho E_\alpha \cap \mathcal{C}\ell_\rho(D\backslash E_\alpha)$ such that $a_\alpha \neq a_\beta$ whenever $\alpha \neq \beta$. Fix $E' \subset D$ such that both E' and $D\backslash E'$ are ρ-dense. Then our assignment is defined for each $a \,\epsilon\, A\backslash D$ by letting $D_a = E_\alpha$ if $a = a_\alpha$ for some α and $D_a = E'$ otherwise.) For each $a \,\epsilon\, A\backslash D$, choose an infinite sequence $\{a_n\colon n \,\epsilon\, \omega\} \subset D$ that ρ converges to a and has infinite intersection with both D_a and $D\backslash D_a$. Finally, let ρ^* be the topology on A generated by the base $\{\{a\} \cup \{a_n\colon n > m\}\colon a \,\epsilon\, A\backslash D, m \,\epsilon\, \omega\} \cup D$.

Define $X = (A,\mathcal{E}|_A)$ and $Y = (C, \mathcal{E} \cup \rho^*)$—i.e., Y is the set C with the topology generated by the subbase $\mathcal{E} \cup \rho^*$. Standard techniques can be used to show that both X and Y are Lindelöf spaces with clopen bases and hence, by theorem 6.2.7 of ref. 5, have covering dimension zero. The proof that the product $X \times Y$ has positive covering dimension is much harder. Briefly, fix a representation of $(C,\mathcal{E})$ as a subspace of the line so that the order $\leq$ is well defined; then show that the function $f\colon X \times Y \to [-\infty,\infty]$ defined by $f(x,y) = \inf\{\sup c | x \leq c \leq y \text{ or } y \leq c \leq x\}$ is continuous, yet $f^{-1}(a)$ and $f^{-1}(b)$ cannot be separated by a clopen set whenever a and b are distinct real numbers. By theorem 6.2.4 of ref. 5, it follows that $\dim(X \times Y) \geq 1$.

RELATED RESULTS

Variations on example 1 are most easily achieved with the following factorization technique. For a topology $\mathcal{S}$ on a set Z, let $\mathcal{K}(\mathcal{S})$ be the set of all topologies $\mathcal{S}^*$ such that

(*i*) $\mathcal{S}^*$ is a locally countable, locally compact refinement of $\mathcal{S}$, and

(*ii*) whenever E is a countable subset of Z, $c\ell_{\mathcal{S}}E\backslash c\ell_{\mathcal{S}^*}E$ is also countable.

We will use $\mathcal{T}$ to denote the usual topology on the unit interval. It was proved in ref. 12 that the continuum hypothesis implies that $\mathcal{K}(\mathcal{T})$ is not empty. Assume X and Y are subsets of I and λ is a first countable topology on $X \times Y$ that refines $\mathcal{T} \times \mathcal{T}$. Let Δ denote the diagonal in $X \times Y$. Then, a modification of Kunen's technique shows that, under the continuum hypothesis, there exist $\sigma \,\epsilon\, \mathcal{K}(\mathcal{J}|_X)$ and $\tau \,\epsilon\, \mathcal{K}(\mathcal{J}|_Y)$ such that $(\sigma \times \tau)|_\Delta \epsilon\, \mathcal{K}(\lambda|_\Delta)$. Thus, we can factor a given topology λ on $X \times Y$ into a topology σ on X and a topology τ on Y. Although $\sigma \times \tau$ may not equal λ, it approximates it on Δ. Moreover, $\sigma \times \tau$ can be constructed to have many additional properties. If λ is perfect, $\sigma \times \tau$ can be made perfect, and if λ is any reasonable topology,

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$\sigma \times \tau$ can be made normal. Variations of the factorization technique can be used to build virtually any properties into our dimension example that are not outlawed by known positive results. For those concerned with perfect normality, an important property in dimension theory, we have:

Example 2: (CH) There exist spaces X and Y such that $\dim(X \times Y) > \dim X + \dim Y$ and X, Y and $X \times Y$ are all perfectly normal and locally compact.

The spaces in example 2 can be derived directly from the factorization technique by letting $X = Y = C$ and $\lambda|_\Delta = \rho$. It should be pointed out that example 2 was actually constructed before example 1 and hence was the first counterexample to the dimension problem. The construction of example 2 uses the continuum hypothesis because it uses Kunen's technique. Przymusiński (13) has eliminated the continuum hypothesis from example 2 (although his spaces are not perfect) by replacing Kunen's technique by a technique of van Douwen that does not require the continuum hypothesis.

The factorization technique also has applications outside of dimension theory. We include two examples here. A *Radon space* is a topological space on which every Borel measure is regular with respect to compact sets—i.e., $\mu(E) = \sup\{\mu(K) | K \subset E, K \text{ is compact}\}$ for each Borel measure μ and each Borel set E. Schwartz (14) has discussed Radon spaces and asks if the countable product of Radon spaces is always a Radon space.

Example 3: (CH) There exists a compact separable Radon space whose square is not a Radon space.

Our last example answers a question of M. Starbird and M. E. Rudin on Dowker spaces. It follows from the factorization technique by letting $X = Y = L$ and $\lambda|_\Delta = \tau$, where L and τ are as in ref. 12. A Dowker space is a normal space whose product with the unit interval is not normal.

Example 4: (CH) The product of two non-Dowker spaces can be a Dowker space.

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1. Nagata, J. (1971) *Gen Top. Appl.* **1,** 65–77.
2. Fillippov, V. (1973) *Dokl. Akad. Nauk SSSR* Tom **209,** No. 4, 805–807 [English translation: (1973), *Soviet Math. Dokl.* Vol. 14, No. 2, 547–550].
3. Morita, K. (1973) *Gen. Top. Appl.* **3,** 125–133.
4. Alexandroff, P. S. (1951) *Uspehi Mat. Nauk (N.S.)* **6,** No. 5 (45), 43–68 [English translation: (1955) *Am. Math. Soc. Trans. Ser. 2* **1,** 1–26].
5. Engelking, R. (1977) *General Topology* (Polish Scientific Publishers, Warsaw, Poland).
6. Gillman, L. & Jerison, M. (1960) *Rings of Continuous Functions* (Van Nostrand, Princeton, NJ).
7. Nagami, K. (1970) *Dimension Theory* (Academic, New York).
8. Nagata, J. (1967) *Proceedings of the Second Prague Topological Symposia, 1966* (Academia, Prague, Czechoslovakia), pp. 259–270.
9. Pasynkov, B. A. (1973) *Dokl. Akad. Nauk. SSSR* Tom **209,** 792–794 [English translation: *Soviet Math. Dokl.* (1973) **14,** 530–533].
10. Pontrjagin, L. S. (1930) *C. R. Acad. Sci.* **190,** 1105–1107.
11. Oxtoby, J. C. (1971) *Measure and Category* (Springer, New York).
12. Juhász, I., Kunen, K. & Rudin, M. E. (1976) *Can. J. Math.* **28,** 998–1005.
13. Przymusiński, T. C. (1977) *Top. Proc.* **2,** No. 1.
14. Schwartz, L. (1973) *Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures* (Oxford University Press, Oxford, England).