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Mathematics

The boundary value problem for maximal hypersurfaces

(general relativity/Dirichlet problem)

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ABSTRACT A spacelike hypersurface (codimension 1) in a Lorentzian manifold is called a maximal surface if it extremizes the hypervolume functional. Although maximal surfaces are superficially analogous to minimal hypersurfaces in Riemannian geometry, their properties can be dramatically different, as can be seen from the validity of Bernstein's theorem in all dimensions [Cheng, S.-Y. & Yau, S.-T. (1976) *Ann. Math.* 104, 407–419]. Here we establish a point of contact between maximal surfaces and minimal surfaces by solving the Dirichlet problem for acausal boundary data but using boundary curvature conditions similar to those of H. Jenkins and J. Serrin [(1968) *J. Reine Angew. Math.* 229, 170–187].

Maximal hypersurfaces play an important role in the constraint equations of general relativity (1–3), in the positive mass theorem,*† and in the turn-around epoch of the universe (4). Yet, general existence theorems on maximal hypersurfaces remain a mystery. Lichnerowicz (1) stated the difficulties both of application of general elliptic methods and of application of Douglas's method to the solution of the maximal surface equation. Only recently has Choquet-Bruhat (5) shown that spacetimes, close to those with maximal hypersurfaces, themselves contain maximal hypersurfaces. As a first step toward existence of maximal hypersurfaces in asymptotically flat Lorentzian manifolds, we solve the Dirichlet problem for acausal boundaries over domains whose boundaries have nonnegative mean curvature.

The three parts of this note contain a discussion of nonparametric surfaces as the general type of spacelike hypersurface, the derivation of the maximal surface equation and its relationship to mean curvature, and the solution to the boundary value problem.

This work is based on a talk I gave at the 1977 Waterloo conference on general relativity.

1. Causal structure of Lorentzian manifolds

Although a Lorentzian structure on a manifold does not generally produce a distance function as does a Riemannian structure, it does provide a separation of tangent vectors into timelike, spacelike, and null vectors, which in turn yields obvious definitions of timelike, spacelike, and null submanifolds. At this stage, the causal structure induced by timelike (causal) paths is immediately apparent. Thus, two points p and q in a Lorentzian manifold M are said to be chronologically (causally) related if p and q can be joined by a timelike (timelike or null) path. The concept of an achronal (acausal) set plays a role dual to the causal structure. A set S in a Lorentzian manifold M is called achronal (acausal) if and only if no two points of S are chronologically (causally) related. Acausal and achronal sets are introduced not only for their value as order-theoretic replacements of spacelike submanifolds, but also to study their domains of dependence (see ref. 6). Note that any timelike (causal) curve can intersect an achronal (acausal) set at most once.

The relationship between spacelike submanifolds and achronal sets is complicated by the fact that spacelike submanifolds need be neither acausal nor achronal. But the next theorem about Minkowski space, denoted by L^{n+1} and with signature $(+, \ldots, +, -)$, tells us that, if a spacelike hypersurface is not acausal, it must have a boundary.

THEOREM 1. *A spacelike hypersurface* M *without boundary that is a closed set in the canonical topology of* L^{n+1} *is acausal.*

Proof: Suppose that p is a point of M and that the future light cone at p intersects M at q. By taking a tubular neighborhood N of M, you can find a curve c from p to q in $N - M$, in particular not cutting M. But the join of the segment pq with c is homotopic to zero, intersecting M an even number of times, while the join of pq with c cuts M once, a contradiction.

The proof of *Theorem 1* is based on chapter 6 of the book of Hawking and Ellis (7), which contains a complete exposition of partial Cauchy surfaces. Compare also section of ref. 8.

Achronal spacelike hypersurfaces have a nice feature. They can be expressed nonparametrically.

THEOREM 2. *An achronal spacelike submanifold* M *in* L^{n+1} *is the graph of a smooth real valued function whose domain of definition is contained in the hyperplane* $x^{n+1} = 0$.

Proof: Intersect M with straight lines parallel to the x^{n+1} axis.

A partial converse to *Theorem 2* comes from the observation that a spacelike nonparametric hypersurface over a convex domain in $\mathbf{R}^n$ is acausal.

When the term acausal hypersurface is used, it will mean a spacelike hypersurface, possibly with boundary and, hence, the graph of a smooth function u defined on an open domain in $\mathbf{R}^n$ with $|Du|^2 = \Sigma(\partial u/\partial x^i)^2 < 1$.

Finally, there is an extension property for acausal spacelike submanifolds B of codimension 2 in L^{n+1}, which will be needed later for the Dirichlet problem for maximal hypersurfaces.

THEOREM 3. *Given an acausal spacelike submanifold* B *of codimension* 2 *in* L^{n+1} *that is compact without boundary, there is an acausal hypersurface* M *in* L^{n+1} *with boundary* B.

Proof: The theorem is true in more general situations and is contained in theorem 1.4 of the Oxford thesis of A. J. Goddard (9).

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* Y. Choquet-Bruhat, A. Fischer, and J. Marsden, "Maximal hypersurfaces and positivity of mass," preprint.
† R. Schoen and S.-T. Yau, "On the proof of the positive mass conjecture in general relativity," preprint.

2. The fundamental equation

Suppose that M is a spacelike hypersurface in L^{n+1} described by the graph of the function $x^{n+1} = u(x)$, with u of class C^1 on an open domain Ω in $\mathbf{R}^n$. The hypervolume or Minkowski n-measure of M, denoted by $A(M)$, is given by the formula

$$A(M) = \int_{\Omega} (1 - |Du|^2)^{1/2}\, dx^1 \ldots dx^n.$$

If Ω is a bounded domain and $u \in C^1(\overline{\Omega})$, then $A(M)$ is finite.

Varying M through spacelike hypersurfaces with the same boundary yields extremals, called maximal hypersurfaces.

THEOREM 4. *A spacelike hypersurface is an extremum for hypervolume if and only if the function* u *satisfies*

$$(1 - |\mathrm{Du}|^2)\, lap\ \mathrm{u} + hess_{\mathrm{u}}(\mathrm{Du}) = 0 \text{ on } \Omega \qquad [1]$$

with $|\mathrm{Du}|^2 < 1$ *and* $\mathrm{hess_u}$ *the quadratic form associated to* $\mathrm{D^2u}$.

Proof: Eq. 1 follows from the Euler equations for variation of the area.

Denoting the Lorentzian structure of L^{n+1} by $\langle , \rangle$, the relationship between Eq. 1 and mean curvature is contained in the following two facts, the first dating back to Lichnerowicz (1).

THEOREM 5. *Let* F *be a smooth function on* L^{n+1} *with timelike gradient. The mean curvature* h *of a level set of* F *is given by the formula*

$$\mathrm{h} = \frac{1}{\mathrm{n}\|\mathrm{DF}\|}\left(dal\ \mathrm{F} + \frac{hess_{\mathrm{F}}(\mathrm{DF})}{-\langle \mathrm{DF},\mathrm{DF}\rangle}\right)$$

with $\|\mathrm{DF}\| = (-\langle \mathrm{DF},\mathrm{DF}\rangle)^{1/2}$.

Proof: Use the Lorentzian version of theorem 4.5 in ref. 10.

Using the wrong-way Schwarz inequality, Lichnerowicz observed that the equation $h = 0$ is elliptic.

COROLLARY 6. *In* L^{n+1}, *the nonparametric surface* (x,u(x)) *is a maximal spacelike surface if and only if*

$$(1 - |\mathrm{Du}|^2)\, lap\ \mathrm{u} + hess_{\mathrm{u}}(\mathrm{Du}) = 0 \text{ and } |\mathrm{Du}| < 1.$$

Proof: Apply Theorem 5 to the zero set of $F(x,t) = t - u(x)$, $x = (x^1, \ldots, x^n)$ and $t = x^{n+1}$.

Defining the differential operator G by the equation

$$G(u) = (1 - |Du|^2)\ \mathrm{lap}\ u + \mathrm{hess}_u(Du),$$

it is easy to see that the symbol of the operator G is given by

$$(1 - |p|^2)|\xi|^2 + \langle p,\xi\rangle^2.$$

As a result,

$$(1 - |p|^2)|\xi|^2 \le (1 - |p|^2)|\xi|^2 + \langle p,\xi\rangle^2 \le |\xi|^2,$$

and so the operator G is elliptic for $|p| < 1$, but clearly not uniformly elliptic. Together with the fact that G is quasilinear, the technique for solving Eq. 1 will be based on the method of Leray and Schauder (11).

3. Existence theorems for boundary value problems

The Dirichlet problem for Eq. 1 can be simply phrased: Given a function φ: $\partial\Omega \to \mathbf{R}$ does there exist a smooth function u:$\overline{\Omega} \to \mathbf{R}$ such that $G(u) = 0$ on $\overline{\Omega}$, $u = \varphi$ on $\partial\Omega$, and $|Du| < 1$ on $\overline{\Omega}$. Saying that u is smooth on $\overline{\Omega}$ means that u is smooth on Ω and all derivatives have continuous extensions to $\overline{\Omega}$. The challenge then is to determine the nature of the function φ and the boundary $\partial\Omega$. In the remainder of this section, we shall on occasion represent G by a matrix $a^{ij} = a^{ij}(Du)$. In the two-dimensional case with $x = x^1$ and $y = x^2$,

$$(a^{ij}) = (a^{ij}(Du)) = \begin{pmatrix} 1 - u_y^2 & u_x u_y \\ u_x u_y & 1 - u_x^2 \end{pmatrix}.$$

The basic difficulty in applying the Leray–Schauder fixed point theory to the solving of this Dirichlet problem lies in keeping $|Du|$ bounded away from one, but nonnegative mean curvature of the boundary of Ω allows the use of the distance to $\partial\Omega$ as a barrier function as in ref 12.

LEMMA 7. *Suppose that* Ω *is a bounded domain in* $\mathbf{R}^n$ *with* C^2*-boundary, whose mean curvature is nonnegative. Let* φ *belong to* $\mathrm{C}^2(\overline{\Omega})$ *with* $\mathrm{c}_1 = max_{\overline{\Omega}}|\mathrm{D}\varphi| < 1$, *then there exists a constant* $\gamma < 1$ *such that any function* u *in* $\mathrm{C}^2(\overline{\Omega})$ *satisfying the Dirichlet problem* $\mathrm{G(u)} = 0$ *on* Ω, $\mathrm{u} = \varphi$ *on* $\partial\Omega$, $|\mathrm{Du}| < 1$ *on* $\overline{\Omega}$ *has the property that* $max_{\overline{\Omega}}|\mathrm{Du}| \le \gamma$. *Moreover,* γ *depends on* φ.

Proof: We sketch the proof. Assume $c_1 > 0$ and consider $u = \varphi + \beta(\mathrm{dist}(x,\partial\Omega))$, in which β is to be determined and x is a point in a collar neighborhood of $\partial\Omega$. To ensure that $|Du| < 1$, it suffices to require $\beta' < 1 - c_1$. Applying G to u and using the hypothesis that mean curvature of $\partial\Omega$ is nonnegative,

$$\begin{aligned} G(u) &= -\beta'(1 - |Du|^2)nh_{\partial\Omega} + a^{ij}(Du)D_{ij}\varphi \\ &\quad + \frac{\beta''}{(\beta')^2} a^{ij}(Du)D_i(u - \varphi)D_j(u - \varphi) \\ &\le a^{ij}(Du)D_{ij}\varphi + \frac{\beta''}{(\beta')^2} a^{ij}(Du)D_i(u - \varphi)D_j(u - \varphi) \end{aligned}$$

(summation convention in effect),

which implies, since the largest eigenvalue of G is unity and $\beta' < 1 - c_1 < 1$, that

$$G(u) \le \frac{\beta''}{(\beta')^2} + \frac{c_2}{1 - c_1},$$

in which $c_2 = \max|D^2u|$ on $\overline{\Omega}$. Note that if c_2 vanishes, φ is a linear solution to $G(u) = 0$, so let us assume that c_2 is positive. Choose β now as

$$\beta(s) = \frac{1}{\mu}(\log(1 + ks)),\ \mu = \frac{c_2}{1 - c_1}.$$

(Compare chapter 13 of ref. 13 or lemma 7 of ref. 14.) Hence, if the radius of the collar is sufficiently small, the function u is an upper barrier and, applying theorem 1 of paragraph 6 in ref. 15,

$$Du \le c_1 + \beta'(0) =: \gamma.$$

Because the choice of k is still free, we can require that $\beta'(0) = (k/\mu) < 1 - c_1$. A lower barrier is constructed in a similar fashion and so

$$|Du| \le c_1 + \beta'(0) = \gamma < 1.$$

The *Lemma* follows by using the maximum principle applied to Du.

Remark: The above lemma can be used to analyze boundary data $\sigma\varphi$, $0 \le \sigma \le 1$ by observing that $\sigma(\varphi + \beta(\mathrm{dist}(,\partial\Omega)))$ is a barrier for $\sigma\varphi$. For $\sigma = 0$, this follows from the maximum principle. If σ is positive, a calculation shows that $k_\sigma = k$ and $\gamma_\sigma \le \sigma(c_1 + (k/\mu)) = \sigma\gamma \le \gamma$.

THEOREM 8. *Assume that* Ω *is a bounded domain in* $\mathbf{R}^n$ *with* C^2*-boundary,* $\partial\Omega$, *whose mean curvature is nonnegative. If* φ *is in* $\mathrm{C}^2(\overline{\Omega})$ *with* $max_{\overline{\Omega}}|\mathrm{D}\varphi| < 1$, *then there is a function* u *in* $\mathrm{C}^2(\overline{\Omega})$ *with* $\mathrm{G(u)} = 0$ *on* Ω, $\mathrm{u} = \varphi$ *on* $\partial\Omega$, *and* $max|\mathrm{Du}| < 1$ *on* $\overline{\Omega}$.

Proof: Given γ as in *Lemma 7*, choose γ_1 so that $\gamma < \gamma_1 < 1$. Now it is easy to see that the fixed points are in a neighborhood such that $\max_{\overline{\Omega}}|Du| \le \gamma < \gamma_1 < 1$.

There are other conditions on the domain Ω that take the place of the boundary-curvature restriction, but these conditions will be discussed in the expanded version of this note. In ref.

(16), D. Bancel has modified the bounded slope condition to solve the Dirichlet problem.

COROLLARY 9. *Suppose that* B *is a closed acausal spacelike submanifold of codimension two in* L^{n+1} *over a boundary* $\partial\Omega$ *in* $\mathbf{R}^n$ *whose mean curvature is nonnegative. Then there exists an acausal maximal hypersurface with boundary* B.

Proof: It follows from *Theorem 3* that we may consider a nonparametric boundary value problem with the maximum of the absolute value of the boundary value strictly less than one. The corollary then follows directly from *Theorem 8*.

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