Proc. Natl. Acad. Sci. USA
Vol. 76, No. 8, pp. 3599–3601, August 1979
Mathematics

Characterization of parallel subtraction

(Hermitian operators/network connection/geometric mean)

W. N. ANDERSON, JR.*, T. D. MORLEY†, AND G. E. TRAPP*

*Department of Statistics and Computer Science, West Virginia University, Morgantown, West Virginia 26506; and
†Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Communicated by R. J. Duffin, May 8, 1979

ABSTRACT **Parallel subtraction is an operation defined on pairs of positive operators. In terms of electrical networks, one may pose the following problem: Given an electrical network, represented by a specified positive operator, determine the set of positive operators which when connected in parallel with the specified operator yield another prescribed operator. The set of solutions of this electrical network problem is shown to have a minimum. The minimum is termed "the parallel difference of the fixed operators," and the operation is termed "parallel subtraction." The parallel difference is used to obtain explicit error estimates for an iteration procedure which approximates the geometric mean of positive operators. This concept of the geometric mean reduces to the square root of the product of the operators if the operators commute. Finally, by using the geometric mean, an operator version of the Gaussian mean is presented.**

In this paper we extend the theory of parallel subtraction of positive operators. Parallel subtraction is the inverse operation to parallel addition, a concept motivated by electrical network theory. This concept was introduced by Anderson and Duffin (1). Subsequently, Anderson and Trapp (2) and Pekarev and Smul'jan (3) extended the theory of parallel addition. Parallel subtraction was first considered in 1972 (4) and further results were reported in 1976 (3).

Our goal is to characterize the parallel difference in terms of the minimal solution to a particular equation. In the next section, we review the relevant material and present our characterization theorem.

We then consider the geometric mean of positive operators. This operation, introduced by Pusz and Wornowicz (5) and Ando,[‡] generalized the scalar geometric mean. We show that the geometric mean may be approximated by using an iteration scheme; using the parallel difference, we obtain explicit error estimates for the iterates.

The iteration procedure for the geometric mean consists of successively computing the arithmetic and the harmonic means of a sequence of positive operators.

Gauss (6) considered a similar iteration scheme, using instead the arithmetic and the geometric means. We show that the Gaussian iteration scheme also determines an operator mean.

PARALLEL SUBTRACTION

A self-adjoint linear operator on a finite dimensional Hilbert space is said to be positive if the quadratic form $\langle Ax,x\rangle \geq 0$ for all vectors x. In terms of Hermitian matrices, a self-adjoint positive matrix is also called Hermitian positive semi-definite. A positive operator may be singular. If A and B are positive operators, we will write $A \geq B$ if $\langle Ax,x\rangle \geq \langle Bx,x\rangle$ for all vectors X. We will also write $A \gg B$ whenever $\langle Ax,x\rangle > \langle Bx,x\rangle$ for all non-zero vectors X in the range of A. This condition requires that A is strictly greater than B whenever A is non-zero, but $A = B = 0$ is allowed on the null space of A. If A is invertible, then the condition $A \gg B$ is equivalent to saying that $A - B$ is positive definite.

The parallel sum of the two positive operators A and B is denoted by $A{:}B$ and, is defined by $A{:}B = A(A + B)^{-1}B$ whenever the inverse exists; the case of a singular $A + B$ term can be handled by taking the limit of a sequence of invertible approximations. This limiting procedure leads to a well-defined positive operator. Parallel addition is an associative, commutative, order-preserving operation. Moreover, the range of $A{:}B$ is the intersection of the ranges of A and B—i.e., $\text{ran}(A{:}B) = \text{ran}(A) \cap \text{ran}(B)$, where $\text{ran}(C)$ denotes the range of the operator C.

If A and C are positive operators, then the operator A may always be written as $A = F + G$, where F and G are the unique positive operators such that $\text{ran}(F) \subset \text{ran}(C)$ and $\text{ran}(G) \cap \text{ran}(C) = 0$. The operator F is called the shorted operator of A and is denoted $S_C(A)$. The operator C itself is not really used in the construction of F and G. The subspace $\text{ran}(C)$ carries all of the relevant information concerning the shorted operator. For any positive operators A, B, and C, the following string of equalities holds

$$S_C(A{:}B) = S_C(A){:}B = A{:}S_C(B) = S_C(A){:}S_C(B).$$

Further properties, including explicit expressions, of the shorted operator are given in refs. 2 and 6.

In ref. 4 we considered parallel subtraction, the inverse operator to parallel addition. We showed that if $A \gg B$ then the equation $A{:}X = B$ can be solved; the condition $A \gg B$ is also necessary. One solution is given by the parallel difference $B \div A$, defined by $B \div A = A(A - B)^{-1}B$; in this formula the indicated inverse need exist only on the range of A. If A is singular, there will be more solutions, as we shall see. Since $A \gg B$ implies that $\text{ran}(B) \subset \text{ran}(A)$, we see that $\text{ran}(B \div A) = \text{ran}(B)$.

THEOREM 1. *Let* A *and* B *be positive operators with* $A \gg B$. *The positive operator* X *is a solution to* $A{:}X = B$ *if and only if* $X = B \div A + Y$, *where* Y *is a positive operator with* $ran(Y) \cap ran(A) = 0$.

Proof: If X is of the specified form, then $S_A(X) = B \div A$. Thus $A{:}X = S_A(A){:}X = A{:}S_A(X) = A{:}(B \div A) = B$.

Conversely, if $A{:}X = B$, one has $B = S_B(B) = S_B(A){:}S_B(X)$. In the equation $B = S_B(A){:}S_B(X)$, all three operators have the

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‡ Ando, T. (1978) *Topics on Operator Inequalities*, preprint.

same range, so there is a unique operator $S_B(X)$. But $B \div A$ is one such operator. Therefore, $S_B(X) = B \div A$, and thus $X = B \div A + Y$, where Y is a positive operator and $\text{ran}(Y) \cap \text{ran}(B) = 0$. If $\text{ran}(B) = \text{ran}(A)$ the proof is finished.

If $\text{ran}(B) \subset \text{ran}(A)$ properly, let $Y = Z + W$, where $Z = S_A(Y)$. It will suffice to show that $Z = 0$. But if $Z \neq 0$ then

$$\begin{aligned}\text{ran}(A{:}X) &= \text{ran}[A{:}S_A(X)] \\ &= \text{ran}[S_A(X)] = \text{ran}(Z) + \text{ran}(B \div A) \\ &= \text{ran}(Z) + \text{ran}(B) \\ &\neq \text{ran}(B).\end{aligned}$$

Since $A{:}X$ and B have different ranges, they are unequal. Therefore, $Z = 0$ and the theorem follows.

q.e.d.

Theorem 1 shows that $B \div A$ is the minimum positive operator X so that $A{:}X = B$. Another method of characterizing the parallel difference in terms of a minimal solution is given in theorem 3.2 of ref. 3.

THE GEOMETRIC MEAN

As a generalization of the scalar case, one may define the arithmetic mean of the operators A and B by $(A + B)/2$; similarly the harmonic mean is $2(A{:}B)$. The arithmetic-harmonic inequality then becomes

LEMMA 1. *Let* A *and* B *be positive operators, then* $(A + B)/2 \geq 2(A{:}B)$. *Moreover, if* $A \gg B$, *then* $(A + B)/2 \gg 2(A{:}B)$.

Proof. Since only the range of A need be considered, we may assume that A is non-singular. Parallel addition is commutative; therefore $A{:}B = A(A + B)^{-1}B = B(A + B)^{-1}A$. Also, in ref. 1 it is shown that $A{:}B$ may also be written in the following two ways

$$A{:}B = A - A(A + B)^{-1}A = B - B(A + B)^{-1}B.$$

By using these four expressions for parallel addition, direct computation shows that $A + B - 4(A{:}B) = (A - B)(A + B)^{-1}(A - B)$ which is positive, and which is positive definite if $A - B$ is positive definite.

q.e.d.

For positive operators A and B, we denote the geometric mean, as defined by Pusz and Wornowicz (5) and Ando,‡ by $A\#B$; we read this "A sharp B." The convenient representation $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ is due to Ando; it follows that the scalar formula $A\#B = (AB)^{1/2}$ holds whenever the operators commute. Ando has also shown that the various means are related by the formulas $(A + B)/2 \geq A\#B \geq 2(A{:}B)$ and $[(A + B)/2]\#[2(A{:}B)] = A\#B$.

We give below a new construction for $A\#B$, using an averaging technique due to Asplund (7) who considered an entirely different problem. In the scalar case, this averaging technique may be reduced to Newton's method for the square root (see ref. 8, problem 33, p. 54).

THEOREM 2. *Let* A *and* B *be positive operators. With* $A_0 = A$ *and* $B_0 = B$ *define for* $i \geq 1$, $A_{i+1} = (A_i + B_i)/2$ and $B_{i+1} = 2(A_i{:}B_i)$ *then the sequences* $\{A_i\}$ *and* $\{B_i\}$ *each converge monotonically to* $A\#B$.

Proof: For $i \geq 1$, the inequality $A_i \geq B_i$ is an instance of *Lemma 1*, because A_i is the arithmetic mean of A_{i-1} and B_{i-1}, and B_i is the harmonic mean of A_{i-1} and B_{i-1}.

Since $A_i \geq B_i$ we have $A_i = (A_i + A_i)/2 \geq (A_i + B_i)/2 = A_{i+1}$, and

$$B_i = 2(B_i{:}B_i) \leq 2(B_i{:}A_i) = B_{i+1}.$$

Therefore, the A_i sequence is monotone decreasing and the B_i sequence is monotone increasing. Since both sequences are bounded, each limit exists. We now show that there is a common limit—in fact, $A_{i+1} - B_{i+1} \leq (A_i - B_i)/2$. First, $A_{i+1} - B_{i+1} = (A_i + B_i)/2 - 2(A_i{:}B_i)$. Second, we have from above that $B_i \leq 2(A_i{:}B_i)$. Now this implies that $B_i/2 - 2(A_i{:}B_i) \leq -B_i/2$ and therefore $A_{i+1} - B_{i+1} \leq (A_i - B_i)/2$.

We now show that the common limit is $A\#B$. Since A_{i+1} and B_{i+1} are the arithmetic and harmonic means of A_i and B_i, using the earlier mentioned result of Ando, we have that $A_{i+1}\#B_{i+1} = A_i\#B_i$ and by induction $A_i\#B_i = A\#B$. Moreover, at each stage $A_{i+1} \geq A_i\#B_i \geq B_{i+1}$. Since the A_i and B_i sequences converge to a common limit, the limit must be $A\#B$.

q.e.d.

COROLLARY 1. *If* $B \leq C$, *then* $A\#B \leq A\#C$.

Proof. Consider the iteration scheme for $A\#B$ and $A\#C$. At each stage the desired inequality holds and thus it must hold in the limit.

q.e.d.

An alternative argument for the monotonicity of convergence in *Theorem 2* may be expressed in terms of ordinary and parallel differences.

COROLLARY 2. *For* $i > 1$, $A_i - A_{i+1} = (A_i - B_i)/2$, *and thus is positive. Moreover, if* $A \gg B$, *then* $B_i \div B_{i+1} = 2(B_i \div A_i)$

Proof: $A_i - A_{i+1} = A_i - (A_i + B_i)/2 = (A_i - B_i)/2$. For the second part, the hypothesis $A \gg B$ ensures that, by *Lemma 1*, $A_1 \gg B_1$. By induction, $A_i \gg B_i$ for all $i > 1$, and thus $B_i \div A_i$ exists. First, we show that $2(B_i \div A_i)$ satisfies $B_{i+1}{:}X = B_i$. In fact

$$\begin{aligned}B_{i+1}{:}2(B_i \div A_i) &= 2(B_i{:}A_i){:}2(B_i \div A_i) \\ &= 2[B_i{:}A_i{:}(B_i \div A_i)] \\ &= 2(B_i{:}B_i) = B_i\end{aligned}$$

Finally, observe that $\text{ran}(B_i \div A_i) = \text{ran}\, B_i$, and thus by *Theorems 1* and *2*, $2(B_i \div A_i) = B_i \div B_{i+1}$.

q.e.d.

For positive operators A and B, Carlin and Noble (9) have considered the expression $A(A^{-1}B)^{1/2}$ in the study of coupled transmission lines. One may show that the Carlin–Noble formula is an equivalent expression for the geometric mean.

Having obtained the geometric mean of two positive operators, it is natural to ask for an operator equivalent of the scalar Gaussian mean. The Gaussian mean, also termed the arithmetic-geometric mean, is defined as the common limit of two sequences. The sequences are generated by successively computing the arithmetic and geometric means. Given a_0 and b_0 positive real numbers, let $a_{i+1} = (a_i + b_i)/2$ and $b_{i+1} = (a_ib_i)^{1/2}$ for $i \geq 1$. Gauss showed that the common limit of these two sequences may be expressed as an elliptic integral (8). The arithmetic geometric averaging procedure is commonly used to evaluate elliptic integrals (see ref. 10).

In forming the b_i sequence to estimate the Gaussian mean, it is necessary to compute the geometric mean of a_{i-1} and b_{i-1}. For positive operators A and B, the geometric mean $A\#B$ is the appropriate expression. Modeling the proof of *Theorem 2*, one may show the existence of a Gaussian mean for positive operators.

THEOREM 3: Let $A = A_0$ and $B = B_0$ be positive operators for $i = 1, 2, \ldots$ Define

$$A_{i+1} = (A_i + B_i)/2$$
$$B_{i+1} = A_i\#B_i.$$

Then the A_i and B_i sequences converge monotonically to the same operator.

This research was partially supported by Grant MCS 77-03650(A02) from the National Science Foundation.

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