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Mathematics

The homogeneity conjecture

(recursion theory/degrees of unsolvability/isomorphisms of cones of degrees)

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ABSTRACT We show that, for any function f in which Kleene's $\mathcal{O}$ is computable, the ordering of Turing degrees (i.e., degrees of difficulty of computation of functions) is not isomorphic to the ordering of degrees of functions from which f is computable. This refutes a well-known conjecture of H. Rogers, Jr., and others.

The basic notion of relative computability—i.e., the mathematical definition of the intuitive concept of one function f being effectively computable given another function g—was introduced by Turing (1). Post (2) dubbed this notion Turing reducibility (written $f \leq_T g$) and considered the associated equivalence classes ($f \leq_T g$ and $g \leq_T f$) called degrees of unsolvability or Turing degrees. The systematic study of the structure of these degrees (under the ordering induced by Turing reducibility) began with Kleene and Post (3). A striking heuristic principal became evident from the early work in this area: All of the basic theory, including proofs, of partial recursive (i.e., effective) functions carries over immediately to partial functions recursive in any function f. Moreover, the proofs of all structure theorems for the degrees $\mathcal{D}$ ordered by $\leq_T$ carry over to $\mathcal{D}(\geq \mathbf{f})$, the degrees above the degree of f, also ordered by $\leq_T$. This phenomenon, now called relativization, led Rogers (ref. 4, p. 261) to formulate the homogeneity conjecture: For any degree $\mathbf{f}$, $\mathcal{D}(\geq\mathbf{f})$ is isomorphic to $\mathcal{D}$. This possibility has since been raised by a number of others, including Sacks (5), Yates (6) (who suggested that it might be independent), and Simpson (7). A much stronger conjecture that there be an isomorphism preserving the jump operator was disproven by Feiner (8) [see also Yates (ref. 9, §5)]. We show that the basic one fails quite badly.

The key ingredient in our proof is the result from Nerode and Shore (ref. 10, theorem 4.10) that any isomorphism φ of $\mathcal{D}$ onto $\mathcal{D}(\geq\mathbf{f})$ must be the identity on a cone—i.e., there is a $\mathbf{g}$, called the base of the cone, such that $\mathbf{g} \leq \mathbf{h} \Rightarrow \varphi(\mathbf{h}) = \mathbf{h}$. It was proved using fairly complicated codings of second-order arithmetic in $\mathcal{D}$ as part of a larger investigation into the structure and theory of the Turing degrees. We will give another proof that avoids those methods and gives a better computation of the base of the cone that we need for our result here. Rather than give the sharpest possible proof we give one that suffices for the main result and uses only relatively simple codings and well-known facts about $\mathcal{D}$.

THEOREM 1. *If φ is an isomorphism of $\mathcal{D}$ onto $\mathcal{D}(\geq\mathbf{f})$, $\mathbf{g} = \varphi^{-1}(\mathbf{f}^{(2)}) \vee \mathbf{0}^{(2)}$ and $\mathbf{h} \geq \mathbf{g}^{(5)} \vee \varphi(\mathbf{g})^{(5)}$, then $\varphi(\mathbf{h}) = \mathbf{h}$.*

Proof: Let $\mathcal{D}(\leq\mathbf{h})$ be the ordering of degrees below $\mathbf{h}$ under $\leq_T$. For any $\mathbf{h} \geq \mathbf{g}$ set $S(\mathbf{h}) = \{\mathbf{x} \geq \mathbf{g} \mid (\exists\mathbf{d})(\mathcal{D}(\leq\mathbf{x})$ is $\mathbf{d}$-presentable and $\mathbf{d}'' \leq \mathbf{h})\}$. (To say that a partial ordering $\mathcal{P}$ is $\mathbf{d}$-presentable means that there is a relation on $\mathbf{N}$ isomorphic to $\mathcal{P}$ and of degree $\mathbf{d}$.) Let $R(\mathbf{h}) = \{\mathbf{x} \geq \mathbf{g} \mid \mathbf{x}^{(5)} \leq \mathbf{h}\}$ and $T(\mathbf{h}) = \{\mathbf{x} \geq \mathbf{g} \mid \mathbf{x} \leq \mathbf{h}\}$. It is obvious that LUB $T(\mathbf{h})$, the least upper bound of $T(\mathbf{h})$, is $\mathbf{h}$. On the other hand, as $\mathbf{h} \geq \mathbf{g}^{(5)}$ there are $\mathbf{x},\mathbf{y} \in R(\mathbf{h})$ with $\mathbf{x} \vee \mathbf{y} = \mathbf{h}$ by Selman (11). [For a simple proof of this result see Jockusch's review of Selman's paper (12).] Thus $\mathbf{h}$ is also the least upper bound for $R(\mathbf{h})$.

We now claim that $R(\mathbf{h}) \subseteq S(\mathbf{h}) \subseteq T(\mathbf{h})$ so that $\mathbf{h}$ is also the least upper bound of $S(\mathbf{h})$. For the first containment just note that by a straightforward Tarski–Kuratowski computation like those in Rogers (ref. 4, §14.3) the ordering $\mathcal{D}(\leq\mathbf{x})$ is always $\mathbf{x}^{(3)}$ presentable. Thus if $\mathbf{x} \geq \mathbf{g}$ and $\mathbf{x}^{(5)} \leq \mathbf{h}$, $\mathbf{x} \in S(\mathbf{h})$. For the second, suppose that $\mathbf{x} \geq \mathbf{g}$ and $\mathbf{x} \nleq \mathbf{h}$. It is easy to construct an x-presentable distributive lattice $\mathcal{L}$ of "lines and diamonds" such that $\mathbf{x}$ is recursive in the double jump of any presentation of $\mathcal{L}$. [It is an increasing string of either successive elements or diamonds—i.e., the lattice $2^{\{0,1\}}$. At the nth position one puts a diamond iff $n \in x$. See for example Yates (ref. 6, §4).] As explained in Yates (ref. 9, §5) the usual embedding theorems for distributive lattices [e.g., that of Lachlan (13)] show that $\mathcal{L}$ is isomorphic to a segment $\{\mathbf{y} \mid \mathbf{a} \leq \mathbf{y} < \mathbf{b}\}$ where $\mathbf{a}^{(2)} = \mathbf{x}$ and $\mathbf{a} < \mathbf{b} < \mathbf{x}$. [Such a degree $\mathbf{a}$ exists by iterating the Friedberg completeness theorem (ref. 4, §13.3).] Thus if $\mathcal{D}(\leq\mathbf{x})$ is $\mathbf{d}$-presentable, $\mathbf{x} \leq \mathbf{d}''$. Because $\mathbf{x} \nleq \mathbf{h}$ we see that $\mathbf{x} \notin S(\mathbf{h})$. This completes the proof that LUB $S(\mathbf{h}) = \mathbf{h}$.

We now consider $\varphi[S(\mathbf{h})]$, the image of $S(\mathbf{h})$ under φ. Clearly $\varphi[S(\mathbf{h})] = \{\mathbf{x} \geq \varphi(\mathbf{g}) \mid \exists\mathbf{d}(\mathcal{D}(\leq\mathbf{x}) \cap \mathcal{D}(\geq\mathbf{f})$ is $\mathbf{d}$-presentable and $\mathbf{d}'' \leq \mathbf{h})\}$. Essentially the same argument as given above with $\mathbf{g}$ replaced by $\varphi(\mathbf{g})$ now shows that LUB $\varphi[S(\mathbf{h})] = \mathbf{h}$. [Here we use the assumption that $\mathbf{h} \geq \varphi(g)^5$.] Because φ is an isomorphism it must carry $\mathbf{h} =$ LUB $S(\mathbf{h})$ to LUB $\varphi[S(\mathbf{h})] = \mathbf{h}$ as required.

This result can be improved by using more complicated codings to give $\mathbf{c}^{(2)} \vee \varphi(\mathbf{c})^{(2)}$ as the base of the cone where $\mathbf{c} = \varphi^{-1}(\mathbf{f}') \vee \mathbf{0}'$.

We can now use some results on minimal covers to get our counterexample to the homogeneity conjecture. Note that $\mathbf{b}$ is a minimal cover of $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and there is no $\mathbf{c}$ strictly between them.

THEOREM 2. *If $\mathbf{f} \geq \mathcal{O}$ (the complete Π^1_1 set), then $\mathcal{D}(\geq\mathbf{f})$ is not isomorphic to $\mathcal{D}$.*

Proof: Suppose $\varphi{:}\mathcal{D} \rightarrow \mathcal{D}(\geq\mathbf{f})$ is an isomorphism. Let $\mathbf{g}$ be as in *Theorem 1*. As explained there, we have a lattice $\mathcal{L}$ embedded as a segment below $\mathbf{g}$ such that if $\mathcal{L}$ is $\mathbf{d}$-presentable $\mathbf{g} \leq \mathbf{d}''$. This $\mathcal{L}$ is also isomorphic to a segment below $\varphi(\mathbf{g})$. Because $\mathcal{D}(\leq\varphi(\mathbf{g}))$ is $\varphi(\mathbf{g})^{(3)}$-presentable we have that $\mathbf{g} \leq \varphi(\mathbf{g})^{(5)}$. Now $\varphi(\mathbf{g}) = \mathbf{f}^{(2)} \vee \varphi(\mathbf{0}^{(2)})$. Because $\varphi(\mathbf{0}^{(2)}) \geq \mathbf{f}$, $\varphi(\mathbf{g}) \leq \varphi(\mathbf{0}^{(2)})^{(2)}$. Thus $\mathbf{g} \leq \varphi(\mathbf{0}^{(2)})^{(7)}$. By *Theorem 1* we see that the base $\mathbf{h}$ of a cone of fixed points is $\leq \varphi(\mathbf{0}^{(2)})^{(12)}$. Because $\mathbf{h} \geq \mathbf{f} \geq \mathcal{O}$, $\mathbf{h}$ is the base of a cone of minimal covers of degrees above $\mathbf{0}^{(2)}$ by Harrington and Kechris (14) relativized (or just add a condition that $\alpha^{(0)} = \overline{\phi^{(2)}}$). Thus $\varphi(\mathbf{h}) = \mathbf{h}$ is a base of a cone of minimal covers of degrees above $\varphi(\mathbf{0}^{(2)})$. By Jockusch and Soare (15) (again relativized) $\varphi(\mathbf{0}^{(2)})^{(12)}$ is not a minimal cover of any degree above $\varphi(\mathbf{0}^{(2)})$. Because our computation shows that $\mathbf{h} \leq \varphi(\mathbf{0}^{(2)})^{(12)}$, we have the desired contradiction.

Note that we need the results of Harrington and Kechris (14) only to pick out $\mathcal{O}$ as our counter example. The mere existence

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of such a cone of minimal covers [as given by Jockusch (16)] would suffice to disprove the homogeneity conjecture. On the other hand, we only need that $\mathbf{f}^{(n)}$ is above the base of such a cone for some n. Finally, the entire proof can be relativized to show that there is no cone of isomorphic cones.

THEOREM 3. For any degree $\mathbf{f}$ there is a $\mathbf{g} > \mathbf{f}$ such that $\mathcal{D}(\geq \mathbf{f})$ is not isomorphic to $\mathcal{D}(\geq \mathbf{g})$. In particular, if for any n $\mathbf{g}^{(n)} \geq \mathcal{O}^{\mathbf{f}}$, then $\mathcal{D}(\geq \mathbf{g}) \not\cong \mathcal{D}(\geq \mathbf{f})$.

Note Added in Proof. We have improved *Theorem 1* to show that $\mathcal{D}(\geq \mathbf{f})$ is not even elementarily equivalent to $\mathcal{D}$.

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