Proc. Natl. Acad. Sci. USA
Vol. 78, No. 3, pp. 1327–1328, March 1981
Mathematics

Euler–Poisson equations on Lie algebras and the N-dimensional heavy rigid body

(Hamiltonian mechanics/completely integrable systems)

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Communicated by Bertram Kostant, November 13, 1980

ABSTRACT **The classical Euler–Poisson equations describing the motion of a heavy rigid body about a fixed point are generalized to arbitrary Lie algebras as Hamiltonian systems on coadjoint orbits of a tangent bundle Lie group. The N-dimensional Lagrange and symmetric heavy top are thereby shown to be completely integrable.**

This note announces the complete integrability of the N-dimensional generalizations of the heavy symmetric and Lagrange top. The equations of motion for the classical heavy rigid body are the Euler–Poisson equations. I show that these equations make sense on any Lie algebra and that they are Hamiltonian on coadjoint orbits of a tangent bundle Lie group. Such equations naturally arise when reducing Hamiltonian systems on semi-simple Lie groups with energy of the form kinetic (of a left-invariant metric) plus a potential invariant under an isotropy subgroup of the adjoint action. The N-dimensional heavy rigid body is by definition the mechanical system whose equations of motion are the Euler–Poisson equations on so(N). It is shown that, whenever these equations are also Hamiltonian on orbits of a certain Lie subalgebra of the Kac–Moody extension of so(N), they necessarily describe the N-dimensional Lagrange or heavy symmetric top, in which cases the problem is completely integrable. The symmetric heavy top is induced from a completely integrable system on sl(N;$\mathbb{C}$) which is the prototype of a whole family of new completely integrable systems on semi-simple Lie algebras.

Orbits of a tangent bundle Lie group as reduced manifolds and the Euler–Poisson equations

Two linear functionals, μ_0, ν_0 on a Lie algebra canonically determine three symplectic manifolds.

(*i*) If G is a Lie group and $\mathcal{G}$ is its Lie algebra, the tangent bundle TG is a Lie group isomorphic to the semidirect product, $S = G \times \mathcal{G}$, of G with the vector Lie group $\mathcal{G}$ under the adjoint action Ad of G on $\mathcal{G}$. The Lie algebra $\mathcal{S}$ of S is the semidirect product of $\mathcal{G}$ with itself regarded as a trivial Lie algebra under the action ad and has thus the bracket defined by $[(\xi_1,\eta_1),(\xi_2,\eta_2)] = ([\xi_1,\xi_2], [\xi_1,\eta_2] + [\eta_1,\xi_2])$. By the Kirillov–Kostant–Souriau theorem, the coadjoint orbit $S\cdot(\mu_0,\nu_0)$ of S in $\mathcal{S}^*$ (the dual of $\mathcal{S}$) is symplectic with form $\omega_{(\mu_0,\nu_0)}$.

(*ii*) Let $\nu_0 \in \mathcal{G}^*$ be fixed, G_{ν_0} be its isotropy subgroup under the coadjoint action of G on $\mathcal{G}^*$, and $\mathcal{G}_{\nu_0}$ be its Lie algebra. The cotangent bundle T^*G is diffeomorphic to $G \times \mathcal{G}^*$ on which G_{ν_0} acts freely and properly (by left translation on the first factor) admitting a momentum map $J{:}G \times \mathcal{G}^* \to \mathcal{G}^*_{\nu_0}$, $J(g,\alpha) = (\mathrm{Ad}^*_{g^{-1}}\alpha)|\mathcal{G}_{\nu_0}$ with no critical points. Let $\mu_0 \in \mathcal{G}^*$ and $\overline{\mu}_0 = \mu_0|\mathcal{G}_{\nu_0}$. By the Marsden–Weinstein reduction theorem (1, 2), the reduced manifold $(J^{-1}(\bar{\mu}_0)/(G_{\nu_0})_{\overline{\mu}_0},\sigma)$ is symplectic, where $(G_{\nu_0})_{\overline{\mu}_0}$ is the isotropy subgroup of the G_{ν_0} coadjoint action at $\overline{\mu}_0$ and the symplectic form σ is naturally induced from the canonical symplectic structure on T*G.

(*iii*) For $\overline{\mu}_0 \in \mathcal{G}^*_{\nu_0}$ the one-form $\alpha_{\overline{\mu}_0}(g) = \overline{\mu}_0\circ T_gR_{g^{-1}}(R_g$ = right translation) on G is right-invariant and $(G_{\nu_0})_{\overline{\mu}_0}$ is left-invariant, thus inducing a one-form on the quotient $G/(G_{\nu_0})_{\overline{\mu}_0}$. Denote by $\hat{\alpha}_{\overline{\mu}_0}$ its pull-back to $T^*[G/(G_{\nu_0})_{\overline{\mu}_0}]$ and form the symplectic manifold $(T^*[G/(G_{\nu_0})_{\overline{\mu}_0}], \omega_0 + d\hat{\alpha}_{\overline{\mu}_0})$, where ω_0 is the canonical symplectic form on $T^*[(G/G_{\nu_0})_{\overline{\mu}_0}]$.

THEOREM 1. Let μ_0, $\nu_0 \in \mathcal{G}^*$, $\overline{\mu}_0 = \mu_0|\mathcal{G}_{\nu_0}$. *(a) The reduced symplectic manifold* $(J^{-1}(\overline{\mu}_0)/(G_{\nu_0})_{\overline{\mu}_0}, \sigma)$ *is a symplectic covering of the coadjoint orbit* $[S\cdot(\mu,\nu_0), \omega_{(\mu,\nu_0)}]$ *and symplectically embeds onto a subbundle over* $G/(G_{\nu_0})_{\overline{\mu}}$ *of* $(T^*(G/(G_{\nu_0})_{\overline{\mu}}), \omega_0 + d\hat{\alpha}_{\overline{\mu}})$, *for any* $\mu \in \mathcal{G}^*$ *satisfying* $\mu|\mathcal{G}_{\nu_0} = \overline{\mu} = \overline{\mu}_0$. *(b) Under the generic assumption* $(G_{\nu_0})_{\overline{\mu}_0} = G_{\nu_0}$ *the three manifolds* $[S\cdot(\mu,\nu_0), \omega_{(\mu,\nu_0)}]$, $[J^{-1}(\overline{\mu}_0)/G_{\nu_0}, \sigma]$, *and* $[T^*(G/G_{\nu_0}), \omega_0 + d\hat{\alpha}_{\overline{\mu}}]$ *are symplectically diffeomorphic for any* $\mu \in \mathcal{G}^*$ *satisfying* $\mu|\mathcal{G}_{\nu_0} = \overline{\mu} = \overline{\mu}_0$.

If $G_{\nu_0} = G$, one recovers the fact that the reduced manifold $J^{-1}(\mu_0)/G_{\mu_0}$ is symplectically diffeomorphic to the coadjoint orbit $G\cdot\mu_0 \subset \mathcal{G}^*$ with the Kirillov–Kostant–Souriau structure.

THEOREM 2. *Let* f,g: $\mathcal{G}^* \to \mathbb{R}$ *satisfy* $\mathrm{ad}(\mathrm{df}(\mu))^*(\mu) = 0$, $\mathrm{ad}(\mathrm{dg}(\mu))^*(\mu) = 0$ *for all* $\mu \in \mathcal{G}^*$. *Let* $f_a(\mu,\nu) = f(\mu + a\nu + a^2\varepsilon)$, $g_b(\mu,\nu) = g(\mu + b\nu + b^2\varepsilon)$, *for* $a,b \in \mathbb{R}$ *arbitrary and* $\varepsilon \in \mathcal{G}^*$ *fixed. Then the Poisson bracket* $\{f_a,g_b\}$ *vanishes on any coadjoint orbit of* S *in* $\mathcal{S}^*$.

Any invariant Hamiltonian on T^*G naturally induces Hamiltonian vector fields on all reduced manifolds and hence on all coadjoint orbits of S. If $\mathcal{G}$ admits a bilinear, symmetric, nondegenerate, biinvariant two-form κ, then κ induces such a form on $\mathcal{S}$ and thus coadjoint and adjoint orbits are diffeomorphic. If, in addition, we are given a left-invariant metric $\langle,\rangle$ on G, $\langle L\cdot,\cdot\rangle = \kappa(\cdot,\cdot)$ and a potential invariant under an isotropy subgroup of the adjoint action of G on $\mathcal{G}$, we get a map V on a specific orbit of G and by reduction we obtain functions $H{:}\mathcal{S} \to \mathbb{R}$ of the form $H(\xi,\eta) = (\tfrac{1}{2})\kappa(L\eta,\eta) + V(\xi)$. Hamilton's equations for H on an arbitrary adjoint orbit of S in $\mathcal{S}$ are the Euler–Poisson equations

$$\dot{\xi} = [\xi, L\eta],\ \dot{\eta} = [\eta, L\eta] + [\xi, (\mathrm{grad}\ V)(\xi)], \qquad [1]$$

where grad is the gradient with respect to κ.

Let $\mathcal{G}$ admit a two-form κ as above. The Lie algebra $\mathcal{S} = \mathcal{G} \times \mathcal{G}$ is diffeomorphic with the invariant (with respect to the Kirillov–Kostant–Souriau structure) submanifold $Q_\varepsilon = \{\xi + \eta h + \varepsilon h^2|\xi,\eta,\varepsilon \in \mathcal{G},\ \varepsilon = \text{fixed}\}$ of the Lie subalgebra $\mathcal{H} = \{\Sigma_{n\geq 0}\,\xi_n h^n|\xi_n \in \mathcal{G}$, finite sum$\}$ in the Kac–Moody extension $\tilde{\mathcal{G}} = \{\Sigma_{n\in \mathbb{Z}}\,\xi_n h^n|\xi_n \in \mathcal{G}$, finite sum$\}$ of $\mathcal{G}$.† $\mathcal{G} \times \mathcal{G}$ thus carries

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† For a quick review of Kac–Moody Lie algebras, see Adler, M. & van Moerbeke, P. (1980) *Adv. Math.* (preprint, Brandeis University, Waltham, MA).

a Poisson bracket and a Hamiltonian vector field homomorphism in general different from the ones induced by the adjoint orbit decomposition of $\mathcal{G}$. Call this structure the modified Euler–Poisson structure and denote by X_f^ε the Hamiltonian vector field for f on $\mathcal{G} \times \mathcal{G}$ induced from Q_ε. Relations between X_f^ε and the usual Hamiltonian vector field X_f on adjoint orbits of $\mathcal{G}$ are called Lenard relations; such relations were first discovered by Lenard for the integrals of the Korteweg–de Vries equation.

The *N*-dimensional Lagrange and symmetric top

The N-dimensional heavy rigid body is defined by the following Euler–Poisson equations on so(N):

$$\dot{\Gamma} = [\Gamma,\Omega],\ \dot{M} = [M,\Omega] + [\Gamma,\chi]. \qquad [2]$$

These are just Eqs. 1 for $V = \kappa(\chi,\cdot)$, χ = constant, $\kappa(\xi,\eta) = -(\frac{1}{2})\,\mathrm{Trace}(\xi\eta)$, $\chi,\xi,\eta \in$ so(N), $L^{-1}(\zeta) = J\zeta + \zeta J$, $J = \mathrm{diag}(J_1, \ldots, J_N)$, $J_i + J_j > 0$ for $i \neq j$, $L(M) = \Omega$. If $N = 3$, [2] are the equations of motion of a heavy rigid body of unit weight about a fixed point (the origin) whose center of mass is χ and has principal moments of inertia $J_2 + J_3, J_1 + J_3, J_1 + J_2$; Ω is the angular velocity of the body, M is its angular momentum, and Γ is the coordinate of the Oz-axis (minus gravitational direction) as viewed by an observer fixed on the body. The Hamiltonian function of [2] is $H(\Gamma,M) = (\frac{1}{2})\kappa(M,\Omega) + \kappa(\Gamma,\chi)$. The Lagrange top is defined by $a = J_1 = J_2$, $b = J_3 = \ldots = J_N$, $\chi_{12} \neq 0$, $\chi_{ij} = 0$ for all $i, j \neq 1, 2$. The symmetric heavy top is defined by $a = b$ and χ arbitrary.

THEOREM 3. *Assume $\chi_{12} \neq 0$. The Euler–Poisson Eqs.* 2 *can be written in the form*

$$(\Gamma + \mathrm{M}h + \mathrm{C}h^2)^{\cdot} = [\Gamma + \mathrm{M}h + \mathrm{C}h^2, \Omega + \chi h] \qquad [3]$$

(the variables are polynomials in h*) if and only if* [2] *defines the motion of the Lagrange or symmetric top and then* $\mathrm{C} = (\mathrm{a} + \mathrm{b})\chi$. *In this case, Eq. 3 is Hamiltonian on the invariant submanifold* $\mathrm{Q_C}$ *in the Lie subalgebra* $\mathcal{H}$ *of the Kac–Moody extension of* so(N).

Denote by $u_{k+1,j}(\Gamma,M)$ [resp. $p_j(\Gamma,M)$] the coefficient of h^j in $[1/(k+1)]\,\mathrm{Trace}(\Gamma + Mh + Ch^2)^{k+1}$ (resp. in $[\det(\Gamma + Mh + Ch^2)]^{1/2}$ if N is even). The dimension of the generic adjoint orbit in the semidirect product so(N) × so(N) is $N(N-1) - 2[N/2]$. *Theorem 2*, the Kostant–Symes involution theorem (3, 4), and some direct computations give the following.

THEOREM 4. *(A) The* $\mathrm{N(N-1)/2 - [N/2]}$ *integrals* $\{u_{k+1,j} | j = 2, \ldots, 2k+1, k = 1, \ldots, N-2, k = odd\}$, *if* N *is odd*, $\{u_{k+1,j}, p_i | j = 2, \ldots, 2k+1, k = 1, \ldots, N-3, k = odd, i = 2, \ldots, N-1\}$, *if* N *is even, are Poisson commuting on* so(N) × so(N) *in both the semidirect product and the modified Euler–Poisson structure bracket. (B) The following Lenard relations hold:*

$$X_{u_{k+1,j}} = -X^{C}_{u_{k+1,j-2}},\ k = odd,\ j = 0, \ldots, 2k+4$$

where $\mathrm{k} = 1, \ldots, \mathrm{N} - 1$ *if* N *is odd and* $\mathrm{k} = 1, \ldots, \mathrm{N} - 2$, *if* N *is even, in which case we have the additional relations*

$$X_{p_i} = -X^{C}_{p_{i-2}},\ i = 0, \ldots, N+2$$

(anything with negative indices is zero).

The symmetric heavy top for χ regular semi-simple is the restriction to so(N) of a completely integrable system on sl(N,ℂ) for which a similar *Theorem 4* holds. The proof of the generic independence of the integrals requires only the structure theory of semi-simple Lie algebras and thus this system represents the prototype of a new family of completely integrable systems on semi-simple Lie algebras.

The generic independence of the integrals for the Lagrange top cannot be proved by the above methods (except for $N = 3, 4$) since the dimension of the generic symplectic leaf in the modified Euler–Poisson structure is strictly bigger than $N(N-1)/2 - [N/2]$ for $N > 4$. The complete integrability of this problem follows from the linearization of the flows on the Jacobian (Prym variety) of the curve $\det(\Gamma + Mh + Ch^2 - zI) = 0$ done with the aid of a modification of the van Moerbeke–Mumford method (5) (which can be applied also to the symmetric top). An extensive study of the three-dimensional case has been described by Ratiu and van Moerbeke.‡

‡ Ratiu, T. & van Moerbeke, P. (1980) Preprint (Brandeis University, Waltham, MA).

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