

# Euler–Poisson equations on Lie algebras and the $N$ -dimensional heavy rigid body

(Hamiltonian mechanics/completely integrable systems)

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Communicated by Bertram Kostant, November 13, 1980

**ABSTRACT** The classical Euler–Poisson equations describing the motion of a heavy rigid body about a fixed point are generalized to arbitrary Lie algebras as Hamiltonian systems on coadjoint orbits of a tangent bundle Lie group. The  $N$ -dimensional Lagrange and symmetric heavy top are thereby shown to be completely integrable.

This note announces the complete integrability of the  $N$ -dimensional generalizations of the heavy symmetric and Lagrange top. The equations of motion for the classical heavy rigid body are the Euler–Poisson equations. I show that these equations make sense on any Lie algebra and that they are Hamiltonian on coadjoint orbits of a tangent bundle Lie group. Such equations naturally arise when reducing Hamiltonian systems on semi-simple Lie groups with energy of the form kinetic (of a left-invariant metric) plus a potential invariant under an isotropy subgroup of the adjoint action. The  $N$ -dimensional heavy rigid body is by definition the mechanical system whose equations of motion are the Euler–Poisson equations on  $\mathfrak{so}(N)$ . It is shown that, whenever these equations are also Hamiltonian on orbits of a certain Lie subalgebra of the Kac–Moody extension of  $\mathfrak{so}(N)$ , they necessarily describe the  $N$ -dimensional Lagrange or heavy symmetric top, in which cases the problem is completely integrable. The symmetric heavy top is induced from a completely integrable system on  $\mathfrak{sl}(N; \mathbb{C})$  which is the prototype of a whole family of new completely integrable systems on semi-simple Lie algebras.

## Orbits of a tangent bundle Lie group as reduced manifolds and the Euler–Poisson equations

Two linear functionals,  $\mu_0, \nu_0$  on a Lie algebra canonically determine three symplectic manifolds.

(i) If  $G$  is a Lie group and  $\mathcal{G}$  is its Lie algebra, the tangent bundle  $TG$  is a Lie group isomorphic to the semidirect product,  $S = G \times \mathcal{G}$ , of  $G$  with the vector Lie group  $\mathcal{G}$  under the adjoint action  $\text{Ad}$  of  $G$  on  $\mathcal{G}$ . The Lie algebra  $\mathcal{S}$  of  $S$  is the semidirect product of  $\mathcal{G}$  with itself regarded as a trivial Lie algebra under the action  $\text{ad}$  and has thus the bracket defined by  $[(\xi_1, \eta_1), (\xi_2, \eta_2)] = ([\xi_1, \xi_2], [\xi_1, \eta_2] + [\eta_1, \xi_2])$ . By the Kirillov–Kostant–Souriau theorem, the coadjoint orbit  $S \cdot (\mu_0, \nu_0)$  of  $S$  in  $\mathcal{S}^*$  (the dual of  $\mathcal{S}$ ) is symplectic with form  $\omega_{(\mu_0, \nu_0)}$ .

(ii) Let  $\nu_0 \in \mathcal{G}^*$  be fixed,  $G_{\nu_0}$  be its isotropy subgroup under the coadjoint action of  $G$  on  $\mathcal{G}^*$ , and  $\mathcal{G}_{\nu_0}$  be its Lie algebra. The cotangent bundle  $T^*G$  is diffeomorphic to  $G \times \mathcal{G}^*$  on which  $G_{\nu_0}$  acts freely and properly (by left translation on the first factor) admitting a momentum map  $J: G \times \mathcal{G}^* \rightarrow \mathcal{G}_{\nu_0}^*$ ,  $J(g, \alpha) = (\text{Ad}_{g^{-1}}^* \alpha)|_{\mathcal{G}_{\nu_0}}$  with no critical points. Let  $\mu_0 \in \mathcal{G}^*$  and  $\bar{\mu}_0 = \mu_0|_{\mathcal{G}_{\nu_0}}$ . By the Marsden–Weinstein reduction theorem (1, 2), the reduced manifold  $(J^{-1}(\bar{\mu}_0)/(G_{\nu_0})_{\bar{\mu}_0}, \sigma)$  is symplectic, where

$(G_{\nu_0})_{\bar{\mu}_0}$  is the isotropy subgroup of the  $G_{\nu_0}$  coadjoint action at  $\bar{\mu}_0$  and the symplectic form  $\sigma$  is naturally induced from the canonical symplectic structure on  $T^*G$ .

(iii) For  $\bar{\mu}_0 \in \mathcal{G}_{\nu_0}^*$  the one-form  $\alpha_{\bar{\mu}_0}(g) = \bar{\mu}_0 \circ T_g R_{g^{-1}}$  ( $R_g =$  right translation) on  $G$  is right-invariant and  $(G_{\nu_0})_{\bar{\mu}_0}$  is left-invariant, thus inducing a one-form on the quotient  $G/(G_{\nu_0})_{\bar{\mu}_0}$ . Denote by  $\hat{\alpha}_{\bar{\mu}_0}$  its pull-back to  $T^*[G/(G_{\nu_0})_{\bar{\mu}_0}]$  and form the symplectic manifold  $(T^*[G/(G_{\nu_0})_{\bar{\mu}_0}], \omega_0 + d\hat{\alpha}_{\bar{\mu}_0})$ , where  $\omega_0$  is the canonical symplectic form on  $T^*[G/(G_{\nu_0})_{\bar{\mu}_0}]$ .

**THEOREM 1.** Let  $\mu_0, \nu_0 \in \mathcal{G}^*$ ,  $\bar{\mu}_0 = \mu_0|_{\mathcal{G}_{\nu_0}}$ . (a) The reduced symplectic manifold  $(J^{-1}(\bar{\mu}_0)/(G_{\nu_0})_{\bar{\mu}_0}, \sigma)$  is a symplectic covering of the coadjoint orbit  $[S \cdot (\mu, \nu_0), \omega_{(\mu, \nu_0)}]$  and symplectically embeds onto a subbundle over  $G/(G_{\nu_0})_{\bar{\mu}_0}$  of  $(T^*(G/(G_{\nu_0})_{\bar{\mu}_0}), \omega_0 + d\hat{\alpha}_{\bar{\mu}_0})$ , for any  $\mu \in \mathcal{G}^*$  satisfying  $\mu|_{\mathcal{G}_{\nu_0}} = \bar{\mu}_0$ . (b) Under the generic assumption  $(G_{\nu_0})_{\bar{\mu}_0} = G_{\nu_0}$  the three manifolds  $[S \cdot (\mu, \nu_0), \omega_{(\mu, \nu_0)}]$ ,  $[J^{-1}(\bar{\mu}_0)/G_{\nu_0}, \sigma]$ , and  $[T^*(G/(G_{\nu_0})_{\bar{\mu}_0}), \omega_0 + d\hat{\alpha}_{\bar{\mu}_0}]$  are symplectically diffeomorphic for any  $\mu \in \mathcal{G}^*$  satisfying  $\mu|_{\mathcal{G}_{\nu_0}} = \bar{\mu}_0$ .

If  $G_{\nu_0} = G$ , one recovers the fact that the reduced manifold  $J^{-1}(\mu_0)/G_{\mu_0}$  is symplectically diffeomorphic to the coadjoint orbit  $G \cdot \mu_0 \subset \mathcal{G}^*$  with the Kirillov–Kostant–Souriau structure.

**THEOREM 2.** Let  $f, g: \mathcal{G}^* \rightarrow \mathbb{R}$  satisfy  $\text{ad}(\text{df}(\mu))^*(\mu) = 0$ ,  $\text{ad}(\text{dg}(\mu))^*(\mu) = 0$  for all  $\mu \in \mathcal{G}^*$ . Let  $f_a(\mu, \nu) = f(\mu + a\nu + a^2\varepsilon)$ ,  $g_b(\mu, \nu) = g(\mu + b\nu + b^2\varepsilon)$ , for  $a, b \in \mathbb{R}$  arbitrary and  $\varepsilon \in \mathcal{G}^*$  fixed. Then the Poisson bracket  $\{f_a, g_b\}$  vanishes on any coadjoint orbit of  $S$  in  $\mathcal{S}^*$ .

Any invariant Hamiltonian on  $T^*G$  naturally induces Hamiltonian vector fields on all reduced manifolds and hence on all coadjoint orbits of  $S$ . If  $\mathcal{G}$  admits a bilinear, symmetric, non-degenerate, biinvariant two-form  $\kappa$ , then  $\kappa$  induces such a form on  $\mathcal{S}$  and thus coadjoint and adjoint orbits are diffeomorphic. If, in addition, we are given a left-invariant metric  $\langle \cdot, \cdot \rangle$  on  $G$ ,  $\langle L, \cdot \rangle = \kappa(\cdot, \cdot)$  and a potential invariant under an isotropy subgroup of the adjoint action of  $G$  on  $\mathcal{G}$ , we get a map  $V$  on a specific orbit of  $G$  and by reduction we obtain functions  $H: \mathcal{S} \rightarrow \mathbb{R}$  of the form  $H(\xi, \eta) = (\frac{1}{2})\kappa(L\eta, \eta) + V(\xi)$ . Hamilton's equations for  $H$  on an arbitrary adjoint orbit of  $S$  in  $\mathcal{S}$  are the Euler–Poisson equations

$$\dot{\xi} = [\xi, L\eta], \quad \dot{\eta} = [\eta, L\eta] + [\xi, (\text{grad } V)(\xi)], \quad [1]$$

where  $\text{grad}$  is the gradient with respect to  $\kappa$ .

Let  $\mathcal{G}$  admit a two-form  $\kappa$  as above. The Lie algebra  $\mathcal{S} = \mathcal{G} \times \mathcal{G}$  is diffeomorphic with the invariant (with respect to the Kirillov–Kostant–Souriau structure) submanifold  $Q_\varepsilon = \{\xi + \eta h + \varepsilon h^2 | \xi, \eta, \varepsilon \in \mathcal{G}, \varepsilon = \text{fixed}\}$  of the Lie subalgebra  $\mathcal{K} = \{\sum_{n=0}^{\infty} \xi_n h^n | \xi_n \in \mathcal{G}, \text{finite sum}\}$  in the Kac–Moody extension  $\mathcal{G} = \{\sum_{n \in \mathbb{Z}} \xi_n h^n | \xi_n \in \mathcal{G}, \text{finite sum}\}$  of  $\mathcal{G}$ .<sup>†</sup>  $\mathcal{G} \times \mathcal{G}$  thus carries

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<sup>†</sup> For a quick review of Kac–Moody Lie algebras, see Adler, M. & van Moerbeke, P. (1980) *Adv. Math.* (preprint, Brandeis University, Waltham, MA).

a Poisson bracket and a Hamiltonian vector field homomorphism in general different from the ones induced by the adjoint orbit decomposition of  $\mathcal{S}$ . Call this structure the modified Euler–Poisson structure and denote by  $X_f^\varepsilon$  the Hamiltonian vector field for  $f$  on  $\mathcal{G} \times \mathcal{G}$  induced from  $Q_\varepsilon$ . Relations between  $X_f^\varepsilon$  and the usual Hamiltonian vector field  $X_f$  on adjoint orbits of  $\mathcal{S}$  are called Lenard relations; such relations were first discovered by Lenard for the integrals of the Korteweg–de Vries equation.

**The  $N$ -dimensional Lagrange and symmetric top**

The  $N$ -dimensional heavy rigid body is defined by the following Euler–Poisson equations on  $\mathfrak{so}(N)$ :

$$\dot{\Gamma} = [\Gamma, \Omega], \dot{M} = [M, \Omega] + [\Gamma, \chi]. \tag{2}$$

These are just Eqs. 1 for  $V = \kappa(\chi, \cdot)$ ,  $\chi = \text{constant}$ ,  $\kappa(\xi, \eta) = -(\frac{1}{2}) \text{Trace}(\xi\eta)$ ,  $\chi, \xi, \eta \in \mathfrak{so}(N)$ ,  $L^{-1}(\zeta) = J\zeta + \zeta J$ ,  $J = \text{diag}(J_1, \dots, J_N)$ ,  $J_i + J_j > 0$  for  $i \neq j$ ,  $L(M) = \Omega$ . If  $N = 3$ , [2] are the equations of motion of a heavy rigid body of unit weight about a fixed point (the origin) whose center of mass is  $\chi$  and has principal moments of inertia  $J_2 + J_3, J_1 + J_3, J_1 + J_2$ ;  $\Omega$  is the angular velocity of the body,  $M$  is its angular momentum, and  $\Gamma$  is the coordinate of the Oz-axis (minus gravitational direction) as viewed by an observer fixed on the body. The Hamiltonian function of [2] is  $H(\Gamma, M) = (\frac{1}{2})\kappa(M, \Omega) + \kappa(\Gamma, \chi)$ . The Lagrange top is defined by  $a = J_1 = J_2, b = J_3 = \dots = J_N, \chi_{12} \neq 0, \chi_{ij} = 0$  for all  $i, j \neq 1, 2$ . The symmetric heavy top is defined by  $a = b$  and  $\chi$  arbitrary.

**THEOREM 3.** Assume  $\chi_{12} \neq 0$ . The Euler–Poisson Eqs. 2 can be written in the form

$$(\Gamma + Mh + Ch^2)^\cdot = [\Gamma + Mh + Ch^2, \Omega + \chi h] \tag{3}$$

(the variables are polynomials in  $h$ ) if and only if [2] defines the motion of the Lagrange or symmetric top and then  $C = (a + b)\chi$ . In this case, Eq. 3 is Hamiltonian on the invariant submanifold  $Q_C$  in the Lie subalgebra  $\mathcal{K}$  of the Kac–Moody extension of  $\mathfrak{so}(N)$ .

Denote by  $u_{k+1,j}(\Gamma, M)$  [resp.  $p_j(\Gamma, M)$ ] the coefficient of  $h^j$  in  $[1/(k + 1)] \text{Trace}(\Gamma + Mh + Ch^2)^{k+1}$  (resp. in  $[\det(\Gamma + Mh + Ch^2)]^{1/2}$  if  $N$  is even). The dimension of the generic adjoint orbit in the semidirect product  $\mathfrak{so}(N) \times \mathfrak{so}(N)$  is  $N(N - 1) - 2[N/2]$ . Theorem 2, the Kostant–Symes involution theorem (3, 4), and some direct computations give the following.

**THEOREM 4.** (A) The  $N(N - 1)/2 - [N/2]$  integrals  $\{u_{k+1,j} | j = 2, \dots, 2k + 1, k = 1, \dots, N - 2, k = \text{odd}\}$ , if  $N$  is odd,  $\{u_{k+1,j}, p_i | j = 2, \dots, 2k + 1, k = 1, \dots, N - 3, k = \text{odd}, i = 2, \dots, N - 1\}$ , if  $N$  is even, are Poisson commuting on  $\mathfrak{so}(N) \times \mathfrak{so}(N)$  in both the semidirect product and the modified Euler–Poisson structure bracket. (B) The following Lenard relations hold:

$$X_{u_{k+1,j}} = -X_{u_{k+1,j-2}}^C, k = \text{odd}, j = 0, \dots, 2k + 4$$

where  $k = 1, \dots, N - 1$  if  $N$  is odd and  $k = 1, \dots, N - 2$ , if  $N$  is even, in which case we have the additional relations

$$X_{p_i} = -X_{p_{i-2}}^C, i = 0, \dots, N + 2$$

(anything with negative indices is zero).

The symmetric heavy top for  $\chi$  regular semi-simple is the restriction to  $\mathfrak{so}(N)$  of a completely integrable system on  $\mathfrak{sl}(N, \mathbb{C})$  for which a similar Theorem 4 holds. The proof of the generic independence of the integrals requires only the structure theory of semi-simple Lie algebras and thus this system represents the prototype of a new family of completely integrable systems on semi-simple Lie algebras.

The generic independence of the integrals for the Lagrange top cannot be proved by the above methods (except for  $N = 3, 4$ ) since the dimension of the generic symplectic leaf in the modified Euler–Poisson structure is strictly bigger than  $N(N - 1)/2 - [N/2]$  for  $N > 4$ . The complete integrability of this problem follows from the linearization of the flows on the Jacobian (Prym variety) of the curve  $\det(\Gamma + Mh + Ch^2 - zI) = 0$  done with the aid of a modification of the van Moerbeke–Mumford method (5) (which can be applied also to the symmetric top). An extensive study of the three-dimensional case has been described by Ratiu and van Moerbeke.†

† Ratiu, T. & van Moerbeke, P. (1980) Preprint (Brandeis University, Waltham, MA).

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