Proc. Natl. Acad. Sci. USA
Vol. 78, No. 6, pp. 3308–3312, June 1981
Mathematics

Spin and wedge representations of infinite-dimensional Lie algebras and groups

(Clifford algebra/spin representation/affine Kac–Moody Lie algebra/highest weight representation/line bundle)

VICTOR G. KAC* AND DALE H. PETERSON†

*Massachusetts Institute of Technology, Cambridge, Massachusetts 02139; and †University of Michigan, Ann Arbor, Michigan 48109

Communicated by Bertram Kostant, November 24, 1980

ABSTRACT We suggest a purely algebraic construction of the spin representation of an infinite-dimensional orthogonal Lie algebra (sections 1 and 2) and a corresponding group (section 4). From this we deduce a construction of all level-one highest-weight representations of orthogonal affine Lie algebras in terms of creation and annihilation operators on an infinite-dimensional Grassmann algebra (section 3). We also give a similar construction of the level-one representations of the general linear affine Lie algebra in an infinite-dimensional "wedge space." Along these lines we construct the corresponding representations of the universal central extension of the group $SL_n(k[t,t^{-1}])$ in spaces of sections of line bundles over infinite-dimensional homogeneous spaces (section 5).

1. Let V be a vector space over a field k with $2 \neq 0$ (we do *not* assume that $\dim V < \infty$) and ϕ be a nondegenerate k-valued symmetric bilinear form on V. Define Lie algebras (with the usual bracket):

$$o(V;\phi) = \{a \in \mathrm{End}_k V \mid \phi(ax,y) + \phi(x,ay) = 0,\ x,y \in V\},$$

$$o_{\mathrm{fin}}(V;\phi) = \{a \in o(V;\phi) \mid \dim a(V) < \infty\}.$$

Suppose U is an isotropic subspace of V satisfying

$$U^{\perp\perp} = U;\ \dim U^{\perp}/U \leq 1. \qquad [1]$$

Introduce a Lie subalgebra of $o(V;\phi)$:

$$o(V,U;\phi) = \{a \in o(V;\phi) \mid \dim\, [U + a(U)]/U < \infty\}.$$

We shall construct the projective spin representation $\sigma_{V,U}$ of the Lie algebra $o(V,U;\phi)$.

Recall (ref. 2) that the *Clifford algebra* $C\ell V$ associated to (V,ϕ) is an associative k-algebra with unit 1, defined as the quotient of the tensor algebra over V by the two-sided ideal generated by elements of the form $x \otimes y + y \otimes x - \phi(x,y)1$, $x,y \in V$. We identify V with a subspace of $C\ell V$. If $a \in o(V;\phi)$, the action of a on V extends uniquely to a derivation $\pi(a)$ of $C\ell V$, and π is a representation of $o(V;\phi)$ on $C\ell V$.

Let $C\ell_2 V$ be the linear span in $C\ell V$ of elements of the form $[x,y]{:} = xy - yx$ for $x,y \in V$. Then $C\ell_2 V$ is a Lie subalgebra of $C\ell V$. If $a \in C\ell_2 V$, $x \in V$, then $[a,x] \in V$, and $x \mapsto [a,x]$ lies in $o_{\mathrm{fin}}(V;\phi)$, allowing us to identify $C\ell_2 V$ and $o_{\mathrm{fin}}(V)$. For $a \in C\ell_2 V$, $x \in C\ell V$, one has $\pi(a)x = [a,x]$. Let $(C\ell V)U$ be the left ideal of $C\ell V$ generated by U, and set $s(V,U) = C\ell V/(C\ell V)U$. For $a \in o(V,U;\phi)$, there exists $\tilde{a} \in C\ell_2 V$ such that

$$\pi(a - \tilde{a})U \subset U. \qquad [2]$$

Furthermore, $\tilde{a}$ is defined modulo $(C\ell V)U + k$ by Eq. 2. This allows us to define the projective *spin representation* $\sigma_{V,U}$ of $o(V,U;\phi)$ on the space $s(V,U)$ by

$$\sigma_{V,U}(a)[x + (C\ell V)U] = \pi(a)x + x\tilde{a} + (C\ell V)U.$$

Note that if $a \in o_{\mathrm{fin}}(V) = C\ell_2 V$, then $\pi(a)x = ax - xa$, and we may take $\tilde{a} = a$, so that we obtain the usual definition (ref. 2): $\sigma_{V,U}(a)[x + (C\ell V)U] = ax + (C\ell V)U$. Hence, on $o_{\mathrm{fin}}(V)$, $\sigma_{V,U}$ is the usual spin representation.

Let α be the involutive automorphism of $C\ell V$ defined by $\alpha(x) = -x, x \in V$ and $C\ell V = C\ell^+ V \oplus C\ell^- V$ be the corresponding eigenspace decomposition. This induces the decomposition $s(V,U) = s^+(V,U) \oplus s^-(V,U)$ into a direct sum of irreducible *half-spin representations*, $\sigma^+_{V,U}$ and $\sigma^-_{V,U}$.

As soon as the choice $a \mapsto \tilde{a}$ is made, we obtain a k-valued two-cocycle γ of the Lie algebra $o(V,U;\phi)$:

$$\gamma(a,b)I_{s(V,U)} = [\sigma_{V,U}(a),\ \sigma_{V,U}(b)] - \sigma_{V,U}([a,b]).$$

To make this choice, suppose that U' is a subspace of U with $\dim U/U' < \infty$ and that U'' is a subspace of V such that we have (nonorthogonal) direct sum decompositions $V = U' \oplus U''$, $V = U'^{\perp} \oplus U''^{\perp}$. Let $p'{:}\ V \to U'$, $p''{:}\ V \to U''$, $p'^{\perp}{:}\ V \to U'^{\perp}$, $p''^{\perp}{:}\ V \to U''^{\perp}$ be the associated projections. Then for $a \in o(V,U;\phi)$, we choose

$$\tilde{a} = p''ap'^{\perp}. \qquad [3]$$

Then we have

$$\gamma(a,b) = \frac{1}{2}\,\mathrm{trace}\ p'(ap''b - bp''a)p'^{\perp}. \qquad [4]$$

Now suppose in addition that $U''^{\perp}$ is isotropic and that there exists a (indexed) subset $\{u_i{:}\ i \in I\} \subset U'$ and a basis $\{u^i{:}\ i \in I\}$ of $U''^{\perp}$ such that $\phi(u_i,u^j) = \delta_{ij}$. Let p_0 be a projection of V along $U' + U''^{\perp}$ onto the finite-dimensional subspace $U'^{\perp} \cap U''$ of V. Then for any $a \in o(V,U;\phi)$, there exists a set $\{a_j{:}j \in J\} \subset C\ell V$ such that for any $x \in C\ell V$, there exists a finite set $J(x) \subset J$ such that $a_j x \in (C\ell V)U$ for $j \notin J(x)$, and $\sigma(a)[x + (C\ell V)U] = \Sigma_{j\in J(x)} a_j x + (C\ell V)U$. In this case, we write $\sigma(a) = \Sigma_{j\in J}\, a_j$. Then for $a \in o(V,U;\phi)$, we have

$$\sigma_{V,U}(a) = \frac{1}{2}(ap_0 + p_0 a) + \frac{1}{2}\sum_{i\in I}(au^i)u_i - u^i(au_i). \qquad [5]$$

2. Let $k[t,t^{-1}]$ be the ring of finite Laurent series over k. For $P = \Sigma_{i\in\mathbf{Z}} c_i t^i \in k[t,t^{-1}]$, set Res $P = c_{-1}$. Let V be a finite dimensional vector space over k, and let $\tilde{V} = k[t,t^{-1}] \otimes_k V = \oplus_{s\in\mathbf{Z}}(t^s \otimes V)$ be the associated *loop space*, regarded as a vector space over k.

Call $A \in \mathrm{End}_k \tilde{V}$ homogeneous of degree m if $A(t^s \otimes V) \subset t^{s+m} \otimes V$ for $s \in \mathbf{Z}$. In this case, we assign to A the sequence $A^{(i)} \in \mathrm{End}_k V$, $i \in \mathbf{Z}$, defined by $A(t^i \otimes x) = t^{i+m} \otimes A^{(i)}(x)$, $x \in V$. Let $gl_*(V)$ be the Lie subalgebra of $\mathrm{End}_k \tilde{V}$ spanned by the homogeneous $A \in \mathrm{End}_k \tilde{V}$.

The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked "*advertisement*" in accordance with 18 U. S. C. §1734 solely to indicate this fact.

Fix $r \in \mathbf{Z}$, and set $r' = [r/2]$. Let ϕ be a nondegenerate k-valued symmetric bilinear form on V. Define a nondegenerate k-valued symmetric bilinear form $\tilde{\phi}_r$ on $\tilde{V}$ by

$$\tilde{\phi}_r(P \otimes x, Q \otimes y) = \operatorname{Res} t^r PQ\phi(x,y),\ P,Q \in k[t,t^{-1}],\ x,y \in V.$$

Set $o_*(V;r) = gl_*(V) \cap o(\tilde{V};\tilde{\phi}_r)$. For $s \in \mathbf{Z}$, define a decomposition

$$\tilde{V} = \tilde{V}_s \oplus \tilde{V}'_s,\ \tilde{V}_s = \bigoplus_{i \geq s} (t^i \otimes V),\ \tilde{V}'_s = \bigoplus_{i<s} (t^i \otimes V). \qquad [6]$$

If r is even, set $\tilde{U}_r = \tilde{V}_{-r'}$; if r is odd, suppose U is an isotropic subspace of V satisfying Eq. 1 and set $\tilde{U}_r = \tilde{V}_{-r'} \oplus (t_{-r'-1} \otimes U)$. Then $\tilde{U}_r$ is a $\tilde{\phi}_r$-isotropic subspace of $\tilde{V}$ satisfying Eq. 1, and $o_*(V;r) \subset o(\tilde{V},\tilde{U}_r;\tilde{\phi}_r)$. Therefore, we may define the projective *spin representation* σ_r of $o_*(V;r)$ on $s(\tilde{V},\tilde{U}_r)$ to be the restriction of $\sigma_{\tilde{V},\tilde{U}_r}$ to $o_*(V;r)$.

Set $\tilde{U}'_r = \tilde{V}_{-r'}$, $\tilde{U}''_r = \tilde{V}'_{-r'}$, and define $\tilde{a}$ by Eq. 3 for $a \in o(\tilde{V},\tilde{U}_r;\tilde{\phi}_r)$. Then the cocycle γ is given on $o_*(V;r)$ by Eq. 4 as follows. If A and B are homogeneous of degrees m and s, respectively, then $\gamma(A,B) = 0$ unless $m + s = 0$, and if $m \geq 0$,

$$\gamma(A,B) = \frac{1}{2}\,\delta_{m,-s}\,\operatorname{trace}_V \sum_{j=0}^{m-1} A^{(j-r'-m)}\,B^{(j-r')}. \qquad [7]$$

Fix a basis $u_1,\ldots,u_n$ of V, and let $u^1,\ldots,u^n$ be the dual basis of V with respect to ϕ. For $a \in \operatorname{End}_k V$, let (a_{ij}), $a_{ij} = \phi(u^i, au_j)$, be the matrix of a. Then, if $A \in o_*(V;r)$ is homogeneous of degree m, Eq. 5 gives the following expression for $\sigma_r(A)$ as a pointwise convergent series of left multiplication operators on $s(\tilde{V},\tilde{U}_r)$:

$$\sigma_r(A) = -\frac{1}{2}\sum_{s\in\mathbf{Z}}\sum_{i,j=1}^{n} A_{ji}^{(s)}\,(t^{-s-r-1}\otimes u^i)\,(t^{s+m}\otimes u_j), \quad \text{if } m \neq 0;$$

$$\sigma_r(A) = -\sum_{s\geq -r'}\sum_{i,j=1}^{n} A_{ji}^{(s)}\,(t^{-s-r-1}\otimes u^i)\,(t^{s}\otimes u_j)$$

$$-\frac{1}{2}\delta_{r-2r',1}\sum_{i,j=1}^{n} A_{ji}^{(-r'-1)}\,(t^{-r'-1}\otimes u^i)\,(t^{-r'-1}\otimes u_j), \qquad [8]$$

if $m = 0$.

3a. We assume in section 3 that k is algebraically closed of characteristic 0.

Given a Lie algebra $\mathfrak{g}$ over k, we define the associated *loop algebra* $\tilde{\mathfrak{g}} = k[t,t^{-1}] \otimes_k \mathfrak{g}$ with the obvious bracket, regarded as a Lie algebra over k. Given a finite-dimensional representation $\nu\colon \mathfrak{g} \to \operatorname{End}_k V$, we define in the obvious way a representation $\tilde{\nu}\colon \tilde{\mathfrak{g}} \to \operatorname{End}_k \tilde{V}$. In particular, the orthogonal loop algebra $\tilde{o}(V;\phi) = \oplus_{s\in\mathbf{Z}} (t^s \otimes o(V;\phi))$ acts on $\tilde{V}$, and because $\tilde{o}(V;\phi) \subset o_*(V,r)$, we may restrict σ_r to $\tilde{o}(V;\phi)$. We obtain a projective representation that becomes a linear representation of the central extension $\hat{o}(V;\phi)$ of $\tilde{o}(V;\phi)$ defined below.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over k, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Δ the set of roots of $\mathfrak{g}$ in $\mathfrak{h}$, W the Weyl group of Δ, Δ_+ a set of positive roots, $\mathfrak{n}_+$ the corresponding maximal nilpotent subalgebra of $\mathfrak{g}$, $\Pi = \{\alpha_1,\ldots,\alpha_\ell\}$ the set of simple roots, θ the highest root, and (,) a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (and, hence, on $\mathfrak{g}^*$ and $\mathfrak{h}^*$). For $\alpha \in \mathfrak{h}^*$ with $(\alpha,\alpha) \neq 0$, define $H_\alpha \in \mathfrak{h}$ by $\beta(H_\alpha) = 2(\beta,\alpha)/(\alpha,\alpha)$ for $\beta \in \mathfrak{h}^*$. The *affine Lie algebra* $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$ is an extension $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus kc$ of the loop algebra $\hat{\mathfrak{g}}$ by a one-dimensional center kc, with bracket

$$[A,B]_{\hat{\mathfrak{g}}} = [A,B]_{\tilde{\mathfrak{g}}} + \frac{1}{2}(\theta,\theta)\operatorname{Res}\left(\frac{dA}{dt},B\right)c, \quad \text{for } A,B \in \tilde{\mathfrak{g}}. \qquad [9]$$

Let $\hat{\mathfrak{h}} = \mathfrak{h} \oplus kc$, with basis $\{h_0 := c - H_\theta;\ h_i := H_{\alpha_i},\ 1 \leq i \leq \ell\}$, called the set of dual simple roots. Define reflections $r_i \in GL(\hat{\mathfrak{h}}^*)$, $0 \leq i \leq \ell$, by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$, and let $\hat{W} \subset GL(\hat{\mathfrak{h}}^*)$ be the group generated by the r_i. We regard W as the subgroup of $\hat{W}$ generated by $r_1,\ldots,r_\ell$. Set $\hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \oplus (tk[t] \otimes \mathfrak{g})$. Finally, define derivations d_s, $s \in \mathbf{Z}$, by $[d_s,c] = 0$, $[d_s,A] = t^{s+1}\,dA/dt$ for $A \in \tilde{\mathfrak{g}}$.

Define fundamental weights $\Lambda_i \in \hat{\mathfrak{h}}^*$ by $\Lambda_i(h_j) = \delta_{ij}$, and set $\hat{\rho} = \sum_{i=0}^{\ell} \Lambda_i$. Let $\hat{P}_+$ be the semigroup generated by the Λ_i. For $\Lambda \in \hat{P}_+$, there exists a unique irreducible $\hat{\mathfrak{g}}$-module $L(\Lambda)$, called the *highest weight module* (ref. 3), admitting a nonzero $v \in L(\Lambda)$ such that $\hat{\mathfrak{n}}_+(v) = 0$ and $h(v) = \Lambda(h)v$ for $h \in \hat{\mathfrak{h}}$. $\Lambda(c)$ is called the *level* of the module $L(\Lambda)$; it is a nonnegative integer and is 0 if and only if $L(\Lambda)$ is the trivial one-dimensional module.

Let V be a vector space over k with basis $u_1,\ldots,u_n$, $n \geq 3$, and let ϕ be the symmetric bilinear form on V defined by $\phi(u_i,u_{n-j+1}) = \delta_{ij}$. Then $\mathfrak{g} = o(V;\phi)$ is identified with the Lie algebra of $n \times n$ matrices skew symmetric with respect to the side diagonal. Let $\mathfrak{h}$ be the set of diagonal matrices in $\mathfrak{g}$ and $\mathfrak{n}_+$ be the set of strictly upper triangular matrices in $\mathfrak{g}$. Set $\ell = [n/2]$, and let U be the $\mathfrak{n}_+$-stable maximal isotropic subspace $ku_1 \oplus \ldots \oplus ku_\ell$ of V.

Let E_{ij} be the standard basis elements of the space of $n \times n$ matrices. Set $(A,B) = 1/2$ trace AB. Then setting $s_i = E_{ii} - E_{n-i+1,n-i+1}$ for $1 \leq i \leq \ell$, the dual simple roots of $\hat{o}(V;\phi)$ are $h_0 := c - s_1 - s_2$; $h_i = s_i - s_{i+1}, 1 \leq i \leq \ell-1$; $h_\ell = s_\ell + s_{n-\ell-1}$. [If $n = 4$, we define $\hat{o}(V;\phi)$ by Eq. 9, where $(\theta,\theta) = 2$, even though $o(V;\phi)$ is not simple.] Then the highest weight modules of $\hat{o}(V;\phi)$ of level one are the $L(\Lambda_i)$, where $i = 0,1,\ell-1,\ell$ if n is even, and $i = 0,1,\ell$ if n is odd. These will be realized in section 3b as half-spin modules.

3b. Let $n \in \mathbf{Z}$, $n \geq 3$, and let $r \in \mathbf{Z}$, $r' = [r/2]$. Set $J_- = \{(i,-s) \mid 1 \leq i \leq n,\ s \in (r+1)/2 + \mathbf{Z};\ s > 0 \text{ or } s = 0, \text{ and } i \geq n - \ell + 1\}$, $J_+ = \{(n-i+1,s) \mid (i,-s) \in J_-\}$, $J = J_- \cup J_+$.

For $(i,s) \in J$, set $\xi_{i,s} = t^{s-\frac{r+1}{2}} \otimes u_i$ if n is even; $\xi_{i,s} = \sqrt{-2}\,(t^{s-\frac{r+1}{2}} \otimes u_i)\,(t^{-\frac{r+1}{2}} \otimes u_{\ell+1})$ if n is odd. Then $\{\xi_{i,s}, \xi_{i',s'}\} = \delta_{i+i',n+1}\,\delta_{s,-s'}1$ where $\{a,b\} = ab + ba$. Let $X_{r,n}$ be the (Grassmann) subalgebra of $C\ell\tilde{V}$ generated by the anticommuting elements $\xi_{i,-s}$, $(i,-s) \in J_-$. Set $X^{\pm}_{r,n} = X_{r,n} \cap C\ell^{\pm}\tilde{V}$. For $(i,-s) \in J_-$, define the *creation operator* $\xi^+_{i,-s}$ and the *annihilation operator* $\xi^-_{i,-s}$ by: $\xi^+_{i,-s}(x) = \xi_{i,-s}\,x$, $\xi^-_{i,-s}(1) = 0$, $\xi^-_{i,-s}(\xi_{i',-s'}x) = \delta_{i,i'}\delta_{s,s'}\,x - \xi_{i',-s'}\xi^-_{i,-s}(x)$ for $x \in X_{r,n}$, $(i',-s') \in J_-$.

If nr is even (respectively nr is odd), then $C\ell\tilde{V} = (C\ell\tilde{V})\tilde{U}_r \oplus X_{r,n}$ (respectively $C\ell^+\tilde{V} = (C\ell^-\tilde{V})\tilde{U}_r \oplus X_{r,n}$). Therefore, we may identify $X^{\pm}_{r,n}$ with $s^{\pm}_{\tilde{V},\tilde{U}_r}$ (respectively $X_{r,n} = X^+_{r,n}$ with $s^+_{\tilde{V},\tilde{U}_r}$). Then for $(i,s) \in J$, the action of left multiplication by $\xi_{i,s}$ on $s_{\tilde{V},\tilde{U}_r}$ (respectively $s^+_{\tilde{V},\tilde{U}_r}$) is given on $X_{r,n}$ by the operator $\xi_i(s)$, where for $(i,-s) \in J_-$, $\xi_i(-s) = \xi^+_{i,-s}$, $\xi_{n-i+1}(s) = \xi^-_{i,-s}$.

The following theorems describe the action of $\hat{o}(V;\phi)$ on the half-spin modules $X^{\pm}_{r,n}$.

THEOREM 1. *Let* n *and* r *be integers with* nr *even and* n $\geq$ 3. *Then* (i) *the following formulas define a linear representation* $\hat{\sigma}_r$ *of* $\hat{o}(V;\phi)$ *on* $X_{r,n}$, *which is a linearization of the projective spin representation* σ_r:

$$\hat{\sigma}_r(c) = I; \quad \textit{for } m \neq 0,$$

$$\hat{\sigma}_r(t^m \otimes a) = \frac{1}{2}\sum_{s\in\mathbf{Z}}\sum_{i,j} a_{ij}\,\xi_i\!\left(-s-\frac{r+1}{2}\right)\xi_{n-j+1}\!\left(s+m+\frac{r+1}{2}\right);$$

$$\hat{\sigma}_r(1 \otimes a) = \sum_{s\geq -r'}\sum_{i,j} a_{ij}\,\xi_i\!\left(-s-\frac{r+1}{2}\right)\xi_{n-j+1}\!\left(s+\frac{r+1}{2}\right)$$

$$+\frac{1}{2}\,\delta_{r,2r'+1}\sum_{i,j} a_{ij}\,\xi_i(0)\,\xi_{n-j+1}(0).$$

(ii) *For* $m \in \mathbf{Z}$, *define operators* D_m *on* $X_{r,n}$ *by*

$$D_m = -\frac{1}{4}\sum_{s\in\mathbf{Z}}\sum_i (2s + m + r + 1)\xi_i\left(-s - \frac{r+1}{2}\right) \times \xi_{n-i+1}\left(s + m + \frac{r+1}{2}\right) \quad \textit{for } m \neq 0;$$

$$D_0 = -\frac{1}{2}\sum_{s\geq -r'}\sum_i (2s + r + 1) \times \xi_i\left(-s - \frac{r+1}{2}\right)\xi_{n-i+1}\left(s + \frac{r+1}{2}\right).$$

Then

$$[D_m, D_{m'}] = (m' - m)\, D_{m+m'} + \frac{n}{24}\delta_{m,-m'}\, m(m^2 - 1 + 3\delta_{r,2r'+1})I;$$

$$[D_m, \hat{\sigma}_r(A)] = \hat{\sigma}_r(d_m(A)) \quad \textit{for } A \in \hat{o}(V;\phi).$$

(iii) *As* $\hat{o}(V;\phi)$-*modules*, $X^+_{r,n} \cong L(\Lambda_0)$, $X^-_{r,n} \cong L(\Lambda_1)$ *for* r *even, and* $X^+_{r,n} \cong L(\Lambda_\ell)$, $X^-_{r,n} \cong L(\Lambda_{\ell-1})$ *for* r *odd.*

In the statement of *Theorem 2*, Σ' denotes a sum to be extended only over summands not involving $\xi_{\ell+1}(0)$, which is undefined.

THEOREM 2. *Let* n *and* r *be odd integers with* $n \geq 3$. *Then* (a) *The following formulas define a linear representation* $\hat{\sigma}^+_r$ *of* $\hat{o}(V;\phi)$ *on* $X_{r,n}$, *which is a linearization of the projective half-spin representation* σ^+_r:

$$\hat{\sigma}^+_r(c) = I; \quad \textit{for } m \neq 0,$$

$$\hat{\sigma}^+_r(t^m \otimes a) = \frac{1}{2}\sum_{s\in\mathbf{Z}}\sum_{i,j}{}' a_{ij}\xi_i\left(-s - \frac{r+1}{2}\right) \times \xi_{n-j+1}\left(s + m + \frac{r+1}{2}\right) + \frac{1}{\sqrt{-2}}\sum_i a_{i,\ell+1}\,\xi_i(m);$$

$$\hat{\sigma}^+_r(1 \otimes a) = \sum_{s\geq -r'}\sum_{i,j} a_{ij}\xi_i\left(-s - \frac{r+1}{2}\right)\xi_{n-j+1}\left(s + \frac{r+1}{2}\right) + \frac{1}{2}\sum_{i,j}{}' a_{ij}\xi_i(0)\xi_{n-j+1}(0) + \frac{1}{\sqrt{-2}}\sum_i{}' a_{i,\ell+1}\xi_i(0).$$

(b) *For* $m \in \mathbf{Z}$, *define an operator* D_m *on* $X_{r,n}$ *by*:

$$D_m = -\frac{1}{4}\sum_{s\in\mathbf{Z}}\sum_i{}' (2s + m + r + 1)\xi_i\left(-s - \frac{r+1}{2}\right) \times \xi_{n-i+1}\left(s + m + \frac{r+1}{2}\right) + \frac{m}{2\sqrt{-2}}\xi_{\ell+1}(m)$$

for $m \neq 0$,

$$D_0 = -\frac{1}{2}\sum_{s\geq r'}\sum_i (2s + r + 1) \times \xi_i\left(-s - \frac{r+1}{2}\right)\xi_{n-i+1}\left(s + \frac{r+1}{2}\right).$$

Then:

$$[D_m, D_{m'}] = (m' - m)D_{m+m'} + \frac{n}{24}\delta_{m,-m'}\, m(m^2 + 2)I;$$

$$[D_m, \hat{\sigma}^+_r(A)] = \hat{\sigma}^+_r(d_m(A)) \quad \textit{for } A \in \hat{o}(V;\phi).$$

(c) *As* $\hat{o}(V;\phi)$-*modules*, $s^-_{\tilde{V},\tilde{U}_r} \cong s^+_{\tilde{V},\tilde{U}_r} \cong X_{r,n} \cong L(\Lambda_\ell)$.

These theorems are proved as follows.

Parts *a* follow immediately from Eqs. **7**, **8**, and **9**.

To prove part *b*, consider the Lie algebra $\mathcal{A}: = k[t,t^{-1},d/dt] \otimes_k gl(V)$ of differential operators on $\tilde{V}$. Let $\mathcal{A}_r = \mathcal{A} \cap o(\tilde{V};\tilde{\phi}_r)$ be the subalgebra of $\tilde{\phi}_r$-skew self-adjoint elements of $\mathcal{A}$, spanned by elements of the form

$$\tilde{d}_{m,\ell,a} = \frac{1}{2}\, t^{\ell+m}\left(\left(\frac{d}{dt}\right)^\ell + t^{-(r+\ell+m)}\left(\frac{d}{dt}\right)^\ell t^{r+\ell+m}\right) \otimes a,$$

where $\phi(ax,y) + (-1)^\ell\phi(x,ay) = 0$ for $x,y \in V$. In particular, $\tilde{d}_m: = \tilde{d}_{m,1,I} = t^{m+1}\, d/dt + 1/2(r + m + 1)t^m$ lie in $\mathcal{A}_r$, and $D_m = \sigma_r(\tilde{d}_m)$. Define a two-cocycle γ_0 on $\mathcal{A}$ by:

$$\gamma_0\left(t^{\ell+m}\left(\frac{d}{dt}\right)^\ell \otimes a,\ t^{\ell'+m'}\left(\frac{d}{dt}\right)^{\ell'} \otimes a'\right) = \frac{1}{2}\,\delta_{m,-m'}(\mathrm{trace}_V\, aa')(-1)^\ell \ell!\ell'!\binom{m+\ell}{\ell+\ell'+1}.$$

Then if $r = 0$, the cocycle γ defined by Eq. **7** is the restriction of γ_0 to $\mathcal{A}_0$, and is given by

$$\gamma_0(\tilde{d}_{m,\ell,a}, \tilde{d}_{m',\ell',a'}) = \frac{1}{4}\,\delta_{m,-m'}\,(\mathrm{trace}_V\, aa') \times [(\ell+\ell')! + (-1)^\ell\ell!\ell'!]\binom{m+\ell}{\ell+\ell'+1}.$$

This and Eq. **8** suffice to verify part *b* for $r = 0$; the general case is similar.

Finally, part *c* is proved by using the so-called Weyl–Kac character formula for $L(\Lambda)$. One checks the equality of the characters by applying the "principal specialization" Φ to both sides and using formula 3.29 from ref. 3.

3c. For $\mathfrak{m}$, a semisimple Lie algebra, we define the affine Lie algebra $\hat{\mathfrak{m}}$ to be the direct sum of the affine Lie algebras associated to the simple summands of $\mathfrak{m}$. Then the notions of section 3a generalize in an obvious way to $\mathfrak{m}$ and $\hat{\mathfrak{m}}$. Given a homomorphism τ: $\mathfrak{m} \to o(V;\phi)$, we obtain an induced homomorphism $\hat{\tau}$:$\hat{\mathfrak{m}} \to \hat{o}(V;\phi)$. Then, using the composition $\hat{\sigma}_{-1} \circ \hat{\tau}$ and some choice of maximal isotropic subspace U of V, we regard $X_V: = X_{-1,\dim V}$ as an $\hat{\mathfrak{m}}$-module, called the *spin module of* $\hat{\mathfrak{m}}$ *associated to* τ.

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$ such that $\mathfrak{k}$ is semisimple and contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then, because $\mathfrak{k}$ preserves the Killing form on $\mathfrak{p}$, we have an inclusion $\mathfrak{k} \subset o(\mathfrak{p})$. Let Δ (respectively $\Delta_{\mathfrak{k}}$) be the sets of roots of $\mathfrak{g}$ (respectively $\mathfrak{k}$) in $\mathfrak{h}$, Δ_+ a set of positive roots, $\Delta_{\mathfrak{k}+} = \Delta_+ \cap \Delta_{\mathfrak{k}}$, $\hat{\rho}$ (respectively $\hat{\rho}_{\mathfrak{k}}$) the sum of the fundamental weights of $\hat{\mathfrak{g}}$ (respectively $\hat{\mathfrak{k}}$), W the Weyl group of $\mathfrak{g}$ in $\mathfrak{h}$, $W^1 = \{w \in W: w(\Delta_+) \supset \Delta_{\mathfrak{k}+}\}$. Choose the maximal isotropic subspace $U = \mathfrak{p}_+: = \mathfrak{p} \cap \mathfrak{n}_+$ of $\mathfrak{p}$.

PROPOSITION 1. *The spin module* $X_{\mathfrak{p}}$ *of* $\hat{\mathfrak{k}}$ *associated to the inclusion* $\mathfrak{k} \subset o(\mathfrak{p})$ *has the decomposition*

$$X_{\mathfrak{p}} \cong \bigoplus_{w\in W^1} L(w(\hat{\rho}) - \hat{\rho}_{\mathfrak{k}}).$$

The highest weight vectors are pure spinors lying in $C\ell\mathfrak{p}$.

The proof is essentially the same as that of the finite-dimensional analogue in ref. 4.

We shall describe in detail the most beautiful case

$$\mathfrak{g} = sp(V \oplus V') \supset sp(V) \oplus sp(V') = \mathfrak{k},$$

where $\mathfrak{p} \cong V \otimes V'$ as a $\mathfrak{k}$-module. Let $\dim V = 2n$, $\dim V' = 2m$. Recall that a *composition* of n into $m + 1$ parts is an $(m + 1)$-tuple $(k_0,\ldots,k_m)$ of nonnegative integers with $k_0 + \ldots + k_m = n$; denote by $P_{n,m}$ the set of such compositions. Then, there is a natural bijection between $P_{n,m}$ and the set of all m-element subsets of $\{1,2,\ldots,m+n\}$, so that taking the complementary subset induces a bijection $\pi \mapsto \tilde{\pi}$ from $P_{n,m}$ to $P_{m,n}$. We label the fundamental weights of $\hat{sp}(U)$, $\dim U = 2\ell$, according to the diagram $\overset{0}{\circ}\Longrightarrow\overset{1}{\circ}\!-\!\!-\; \ldots \;-\!\!-\overset{\ell-1}{\circ}\Longleftarrow\overset{\ell}{\circ}$. For $\pi = (k_0,\ldots,k_m) \in P_{n,m}$, so that $\tilde{\pi} = (k'_0,\ldots,k'_n) \in P_{m,n}$, set

$$\Lambda(\pi) = k_0\Lambda_0 + \ldots + k_m\Lambda_m, \Lambda'(\tilde{\pi}) = k'_0\Lambda'_0 + \ldots + k'_n\Lambda'_n.$$

PROPOSITION 2. *The spin module* $X_{V\otimes V'}$ *associated to* $sp(V) \oplus sp(V') \subset o(V \otimes V')$ *has the following* $\hat{sp}(V) \oplus \hat{sp}(V')$*-module decomposition*:

$$X_{V\otimes V'} \cong \bigoplus_{\pi \in P_{n,m}} L(\Lambda(\pi)) \otimes L(\Lambda'(\tilde{\pi})).$$

We note that as a special case of *Proposition 2* we obtain the decomposition of the $\hat{sl}_2 \oplus \hat{sp}_{2n}$-module:

$$X_{-1,4n} \cong \bigoplus_{s=0}^{n} L(s\Lambda_0 + (n-s)\Lambda_1) \otimes L(\Lambda'_s).$$

Remarks. (*a*) One gets similar decompositions for the spin module associated to $o(V) + o(V') \subset o(V \otimes V')$; (*b*) *Proposition 1* can be generalized to the case of $\mathfrak{k}$ and $\mathfrak{g}$ of equal rank, for instance

$$\mathfrak{g} = sl(V \oplus V') \supset sl(V \oplus V') \cap (gl(V) \oplus gl(V')) = \mathfrak{k}.$$

(*c*) The spin representation associated to the adjoint representation of a simple Lie algebra $\mathfrak{g}$ is decomposed into a direct sum of several copies of the $\hat{\mathfrak{g}}$-module $L(\hat{\rho})$. This holds for an arbitrary Kac–Moody Lie algebra and follows from the obvious product decomposition for the character of $L(\rho)$.

4a. Let the assumptions on k, V, ϕ, U be as in section 1. Set $O(V;\phi) = \{g \in GL(V) | \phi(gx,gy) = \phi(x,y)$ for $x,y \in V\}$. For $g \in O(V;\phi)$, extend the action of g on V to an automorphism $\beta \mapsto g \cdot \beta$ of $C\ell V$. Set

$$O(V,U;\phi) = \{g \in O(V;\phi) \mid \dim\,(U + g(U))/U < \infty\}.$$

Then, one can show that $O(V,U;\phi)$ is a subgroup of $O(V,\phi)$. Define a map det: $O(V,U;\phi) \rightarrow \{\pm 1\}$ by

$$\det g = \exp \pi i \dim\,[(U + (I - g)U^{\perp})/U].$$

One can show that this is a homomorphism. Let $SO(V,U;\phi)$ be the kernel of det. We shall define the projective spin representation of the group $O(V,U;\phi)$.

Let $\mathfrak{M}$ be the set of all isotropic subspaces A of V with $A^{\perp\perp} = A$ and $\dim(A+U)/A = \dim(A+U)/U < \infty$. For $A,B \in \mathfrak{M}$ set

$$\langle A|B\rangle_{\pm 1} = \{\bar{\beta} = \beta + (C\ell V)B \mid \beta \in C\ell^{\pm}V, A\beta \subset (C\ell V)B\}.$$

Then one can show (cf. ref. 2) that $\langle A|B\rangle_{\pm 1}$ is a subspace of $C\ell V/(C\ell V)B$ of dimension at most 1. Set

$$\hat{O}(V,U;\phi) = \{(g,\bar{\beta}) \mid g \in O(V,U;\phi),\ \bar{\beta} \in \langle gU|U\rangle_{\det g},\ \bar{\beta} \neq 0\}.$$

This is a group, with multiplication $(g,\bar{\beta})\ (g',\bar{\beta}') = (gg', \overline{(g\cdot\beta')\beta})$; moreover, it is a central extension of $O(V,U;\phi)$ by k^*. Define the *spin representation* $\Sigma_{V,U}$ of the group $\hat{O}(V,U;\phi)$ on the space $s_{V,U}$ by

$$\Sigma_{V,U}((g,\bar{\beta}))[x + (C\ell V)U] = (g \cdot x)\beta + (C\ell V)U.$$

Set $\hat{SO}(V,U;\phi) = \{(g,\bar{\beta}) \in \hat{O}(V,U;\phi) | \det g = 1\}$. Then, it is easy to see that if $U^{\perp} = U$, $s_{V,U}$ is an irreducible $\hat{O}(V,U;\phi)$-module, which decomposes under $\hat{SO}(V,U;\phi)$ into a direct sum $s^+_{V,U} \oplus s^-_{V,U}$ of inequivalent irreducible half-spin modules. If $U^{\perp} \neq U$, then $O(V,U;\phi) = \{\pm I\} \times SO(V,U;\phi)$, and $s_{V,U}$ decomposes under $\hat{SO}(V,U;\phi)$ into a direct sum $s^+_{V,U} \oplus s^-_{V,U}$ of equivalent irreducible half-spin modules.

Remarks. (*a*) Suppose for simplicity that $U^{\perp} = U$. Then for $A,B \in \mathfrak{M}$, the map $u_1 \wedge \ldots \wedge u_n \mapsto u_1 \ldots u_n + (C\ell V)B$ from $\Lambda^{\max}(A/A \cap B)$ to $\langle A|B\rangle$ is an isomorphism. Let $A_i \in \mathfrak{M}$, $i \in \mathbf{Z}/3\mathbf{Z}$. Then by considering the alternating two-form $\tilde{\phi}$ on $A_1 \oplus A_2 \oplus A_3$ given by $\tilde{\phi}(x_1 \oplus x_2 \oplus x_3, y_1 \oplus y_2 \oplus y_3) = \Sigma_i \phi(x_i,y_{i+1}) - \phi(y_i,x_{i+1})$, one obtains a canonical element of $\mathrm{Hom}_k(\langle A_1|A_2\rangle \otimes \langle A_2|A_3\rangle, \langle A_1|A_3\rangle)$, which is the product $\bar{\beta} \otimes \bar{\gamma} \mapsto \bar{\beta\gamma}$. (*b*) $\hat{O}(V,U;\phi)$ is a double cover of a corresponding subgroup of $\hat{GL}(V)$, which is constructed in section 5c. (*c*) Results similar to those of section 5 hold for the spin representations of $o(V,U;\phi)$ and $\hat{O}(V,U;\phi)$. (*d*) One can write down the representation $\Sigma_{V,U}$ and the corresponding cocycle in terms of the Fermi integral (cf. ref. 1). (*e*) One can show that $\Sigma_{V,U} \otimes \Sigma_{V,U}$ is isomorphic to $\Lambda_{V,U}$ (see section 5a) restricted to $O(V,U;\phi)$, provided that $U^{\perp} = U$.

4b. Recall the notation of section 2. Let G be an algebraic group over k. The group $\tilde{G} := G(k[t,t^{-1}])$ is called the associated *loop group*. Given a representation ν: $G \rightarrow \mathrm{End}_k V$, one associates the representation $\tilde{\nu} : \tilde{G} \rightarrow \mathrm{End}_k \tilde{V}$. In particular, we have the general linear and the orthogonal loop groups $\widetilde{GL}(V)$ and $\tilde{O}(V)$. Fix $r \in \mathbf{Z}$. Then $\tilde{O}(V) \in O(\tilde{V},\tilde{U}_r;\tilde{\phi}_r)$, so that we may restrict $\Sigma_{\tilde{V},\tilde{U}_r}$ to $\tilde{O}(V)$, obtaining a projective representation Σ_r of $\tilde{O}(V)$ with differential σ_r.

From *Theorems 1* and *2c* we deduce:

PROPOSITION 3. *The projective representations* $\sigma_r^{\pm}$ *of the group* $SO_n(k[t,t^{-1}])$ *in* $s^{\pm}(\tilde{V},\tilde{U}_r)$ *are irreducible, provided that* char $k = 0$ *and* $n \geq 3$.

5a. Let V be a vector space over a field k (we do not assume that $\dim V < \infty$). Let $\Lambda(V) = \oplus_{k=0}^{\infty} \Lambda^k(V)$ be the exterior algebra over V. If $\dim V = n < \infty$, set $\Lambda^{\max}(V) = \Lambda^n(V)$. For subspaces $A \subset B$ of V with $\dim A < \infty$, we have a canonical inclusion $\Lambda(B/A) \otimes \Lambda^{\max}(A) \subset \Lambda(B)$; if also $\dim B < \infty$, this gives a canonical isomorphism $\Lambda^{\max}(B/A) \otimes \Lambda^{\max}(A) \cong \Lambda^{\max}(B)$.

Two subspaces A,B of V are called commensurable if $\dim(A+B)/(A \cap B) < \infty$; in this case, set $(A|B) = \mathrm{Hom}_k(\Lambda^{\max}(A/A \cap B), \Lambda^{\max}(B/A \cap B))$. For A,B,C, commensurable, $\alpha \in (A|B)$, $\beta \in (B|C)$, define $\alpha\beta \in (A|C)$ by using the canonical isomorphism $(A|B) \otimes (B|C) \cong (A|C)$. Then $(\alpha\beta)\gamma = \alpha(\beta\gamma)$. For A,B, commensurable and $g \in GL(V)$, we have an obvious isomorphism $\alpha \mapsto g \cdot \alpha$ from $(A|B)$ onto $(g(A)|g(B))$. Then $g \cdot (\alpha\beta) = (g \cdot \alpha)(g \cdot \beta)$, $(gg') \cdot \alpha = g \cdot (g' \cdot \alpha)$.

Fix a nonempty family $\mathfrak{M}$ of pairwise commensurable subspaces of V, such that if $A,B \in \mathfrak{M}$, then $A \cap B \in \mathfrak{M}$, and if $A \in \mathfrak{M}$ and $B \supset A$ is a subspace of V with $\dim B/A < \infty$, then $B \in \mathfrak{M}$. Fix $U \in \mathfrak{M}$. Then for $A,B \in \mathfrak{M}$ with $A \subset B$, we have a canonical inclusion $\Lambda(V/B) \otimes (U|B) \subset \Lambda(V/A) \otimes (U|A)$, which allows us to define as an inductive limit the vector space

$$\Lambda(V,U;\mathfrak{M}) = \bigcup_{A \in \mathfrak{M}} \Lambda(V/A) \otimes (U|A).$$

Define a grading $\Lambda(V,U;\mathfrak{M}) = \oplus_{k\in\mathbf{Z}} \Lambda_{(k)}(V,U;\mathfrak{M})$ by the following: if $k \in \mathbf{Z}_+$, $A \in \mathfrak{M}$, $k' = k + \dim A/(A \cap U) - \dim U/(A \cap U)$; then $\Lambda^k(V/A) \otimes (U|A) \subset \Lambda_{(k')}(V,U;\mathfrak{M})$. Then $\Lambda(V,U;\mathfrak{M})$ is a graded $\Lambda(V)$-module.

Let $GL(V;\mathfrak{M})$ be the group of all $g \in GL(V)$ so that g,g^{-1} preserve the family $\mathfrak{M}$. For $U \in \mathfrak{M}$, set $\hat{GL}(V,U;\mathfrak{M}) = \{(\alpha,g)| g \in GL(V;\mathfrak{M}),\ \alpha \in (U|gU),\ \alpha \neq 0\}$, which is a group with product $(\alpha,g)(\alpha',g') = (\alpha(g \cdot \alpha'), gg')$. Then $\hat{GL}(V,U;\mathfrak{M})$ is a central extension of $GL(V;\mathfrak{M})$ by k^*.

If $U' \in \mathfrak{M}$, the isomorphism $(U'|U) \otimes \Lambda(V,U;\mathfrak{M}) \simeq \Lambda(V,U';\mathfrak{M})$ induces a map $(U'|U) \rightarrow \mathrm{Hom}_k(\Lambda(V,U;\mathfrak{M}), \Lambda(V,U';\mathfrak{M}))$, denoted by $\alpha \mapsto \alpha_*$. In particular, this gives a canonical isomorphism $\phi_{U,U'}$: $\hat{GL}(V,U;\mathfrak{M}) \rightarrow \hat{GL}(V,U';\mathfrak{M})$. Finally, for $g \in GL(V;\mathfrak{M})$ we have an obvious isomorphism g_* : $\Lambda(V,U;\mathfrak{M}) \rightarrow \Lambda(V,gU;\mathfrak{M})$. Now we can define the *wedge representation* $\Lambda_{V,U}$ of the group $\hat{GL}(V,U;\mathfrak{M})$ on the space $\Lambda(V,U;\mathfrak{M})$ by $\Lambda_{V,U}(\alpha,g) = \alpha_* \circ g_*$.

For $\alpha \in (U'|U)$, $\hat{g} \in \hat{GL}(V,U;\mathfrak{M})$, $\Lambda_{V,U'}(\phi_{U,U'}(\hat{g})) \circ \alpha_* = \alpha_* \circ \Lambda_{V,U}(\hat{g})$. Therefore, the projective wedge representation of $GL(V;\mathfrak{M})$ on $\Lambda(V,U;\mathfrak{M})$ depends only on V and $\mathfrak{M}$.

Define a subgroup of $GL(V;\mathfrak{M})$ (which is independent of the choice of $U \in \mathfrak{M}$) by

$$GL_0(V;\mathfrak{M}) = \{g \in GL(V;\mathfrak{M}) \mid \dim (U + g(U))/U = \dim (U + g^{-1}(U))/U\}.$$

Then the corresponding subgroup $\hat{GL}_0(V,U;\mathfrak{M}) \subset \hat{GL}(V,U;\mathfrak{M})$ consists of the $\hat{g}$ such that $\Lambda_{V,U}(\hat{g})$ is degree-preserving on $\Lambda(V,U;\mathfrak{M})$.

Set $P = \{U' \in \mathfrak{M} | \dim U/(U \cap U') = \dim U'/(U \cap U')\}$, and define a line bundle $\mathfrak{L}$ on P by $\mathfrak{L} = \{(\alpha',U')| U' \in P, \alpha' \in (U|U')\}$. Then $GL_0(V;\mathfrak{M})$ acts transitively on P, and $\hat{GL}_0(V,U;\mathfrak{M})$ acts on $\mathfrak{L}$ by $(\alpha,g) \cdot (\alpha',U') = (\alpha(g \cdot \alpha'), g(U'))$. Let $\mathfrak{L}^* = \{(\alpha',U')| U' \in P, \alpha' \in (U'|U)\}$ be the dual line bundle.

For $A,B \in \mathfrak{M}$, $A \subset B$, set $P_{A,B} = \{U' \in P|A \subset U' \subset B\}$. Then we regard $P_{A,B}$ as a Grassmanian, thus as a smooth algebraic variety. Then the restrictions of $\mathfrak{L},\mathfrak{L}^*$ to $P_{A,B}$ are algebraic varieties. We call a section of $\mathfrak{L}^*$ over P *regular* if its restriction to each $P_{A,B}$ is a regular map, and denote by $H^0(P,\mathfrak{L}^*)$ the space of regular sections of $\mathfrak{L}^*$ over P. Then the obvious map $\mathfrak{L} \to \Lambda_{(0)}(V,U;\mathfrak{M})$ induces a linear map $\Lambda_{(0)}(V,U;\mathfrak{M})^* \to H^0(P,\mathfrak{L}^*)$. It is easy to see that this map is an isomorphism of $\hat{GL}_0(V,U;\mathfrak{M})$-modules.

5b. Let $d\Lambda$ denote the natural representation of the Lie algebra $gl(V)$ in $\Lambda(V)$. Define the Lie algebra $gl(V;\mathfrak{M}) = \{a \in gl(V)|$ for any $A \in \mathfrak{M}$ there exists $B \in \mathfrak{M}$ such that $A \supset B + a(B)\}$. Let $a \in gl(V;\mathfrak{M})$, $A \in \mathfrak{M}$. Choose $B \in \mathfrak{M}$, with $A \supset B + a(B)$, and a decomposition $V = B \oplus B'$. Consider the sequence of maps: $\Lambda(V/A) \otimes (U|A) \subset \Lambda(V/B) \otimes (U|B) \simeq \Lambda(B') \otimes (U|B) \xrightarrow{d\Lambda(a) \otimes 1} \Lambda(V) \otimes (U|B) \to \Lambda(V/B) \otimes (U|B)$. Their composition defines a map $a_*^A : \Lambda(V/A) \otimes (U|A) \to \Lambda(V,U;\mathfrak{M})$. Up to addition of scalars times the inclusion, a_*^A is independent of the choices of B,B'. From this we may define a degree-preserving projective representation $a \mapsto a_* = \lim_{A \in \mathfrak{M}} a_*^A$ of $gl(V;\mathfrak{M})$ on $\Lambda(V,U;\mathfrak{M})$.

To make a definite choice of a_*, choose a direct sum decomposition $V = U \oplus U'$, with associated projections $p: V \to U$, $p' : V \to U'$. Then we require $a_*(\Lambda^0(V/U) \otimes (U|U)) \subset U'\Lambda(V,U;\mathfrak{M})$, where the right-hand side is defined by using the action of $\Lambda(V)$ on $\Lambda(V,U;\mathfrak{M})$. We define the projective wedge representation $d\Lambda_{V,U}$ of $gl(V;\mathfrak{M})$ on $\Lambda(V,U;\mathfrak{M})$ by $d\Lambda_{V,U}(a) = a_*$. One can show that its two-cocycle is given by (cf. Eq. 4): $\gamma(a,b) = \text{trace } p(ap'b - bp'a)p$.

The representations $\Lambda_{V,U}$ and $d\Lambda_{V,U}$ satisfy the following relations. Let $a \in gl(V;\mathfrak{M})$, $g \in GL(V;\mathfrak{M})$, $\hat{g} = (\alpha,g) \in \hat{GL}(V,U;\mathfrak{M})$, and set $\tilde{p} = g^{-1}pg$. Then

$$\Lambda_{V,U}(\hat{g})d\Lambda_{V,U}(a)\Lambda_{V,U}(\hat{g}^{-1}) = d\Lambda_{V,U}(gag^{-1}) + [\text{trace } (\tilde{p} - p)(a\tilde{p} + pa)]I.$$

5c. Let k be a field of characteristic 0, $V = ku_1 \oplus \ldots \oplus ku_n$, a vector space over k. Recall the notation of section 2.

Regard k as a commutative Lie algebra, so that $\tilde{k} = k[t,t^{-1}]$. Define the affine Lie algebra $\hat{k} = \tilde{k} \oplus kc'$, where c' centralizes $\tilde{k}$ and $[p,q] = (\text{Res } dp/dt\, q)c'$ for $p,q \in \tilde{k}$. Writing $gl(V) = sl(V) \oplus kI$, we set: $\hat{gl}(V) = \hat{sl}(V) \oplus \hat{k}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $sl(V)$, and set $\hat{\mathfrak{h}} = \mathfrak{h} \oplus kc \oplus kI \oplus kc' \subset \hat{gl}(V)$. Define $\Lambda_0', \Lambda_0'' \in \hat{\mathfrak{h}}^*$ by $\Lambda_0'(\mathfrak{h} \oplus kc) = 0 = \Lambda_0''(\mathfrak{h} \oplus kc)$, $\Lambda_0'(I) = 0 = \Lambda_0''(c')$, $\Lambda_0'(c') = 1 = \Lambda_0''(I)$. Then we have a theory of irreducible highest weight modules $L(\Lambda)$ of $\hat{gl}(V)$ (cf. section 3a).

Let $U = k[t] \otimes_k V \subset \tilde{V}$, and let $\mathfrak{M}_0$ be the set of subspaces A of $\tilde{V}$ such that for some $k \in \mathbf{Z}$, $t^{-k}U \supset A \supset t^kU$. Then it is easy to see that $\hat{gl}(V) \subset gl(V;\mathfrak{M}_0)$.

THEOREM 3. *For* $0 \le i \le n-1$, *the* $\hat{gl}(V)$-*modules* $\Lambda_{(i)}(\tilde{V},U;\mathfrak{M}_0)$ *and* $L(\Lambda_i + n\Lambda_0' + i\Lambda_0'')$ *are isomorphic.*

5d. Now we again turn to the corresponding group. Set $U_0 = k[t^{-1}] \otimes_k V$, and let $\mathfrak{M}_0'$ be the set of subspaces A of $\tilde{V}$ such that for some $k \in \mathbf{Z}$, $t^kU_0 \supset A \supset t^{-k}U_0$. Then $\tilde{SL}(V) \subset GL_0(\tilde{V};\mathfrak{M}_0')$, so that we obtain a central extension $\hat{SL}(V) \subset GL_0(V,U_0;\mathfrak{M}_0')$ of $\tilde{SL}(V)$. Then for $j > 0$, there is a unique inclusion $H_j := \{g \in \tilde{SL}(V)| g \equiv I \bmod t^jk[t]\} \subset \hat{SL}(V)$ such that $\Lambda_{\tilde{V},U_0}(a)$ is locally unipotent for $a \in H_j, j > 0$; a subgroup of $\hat{SL}(V)$ is called *open* if it contains H_j for some $j > 0$.

Fix $i \in \mathbf{Z}$, $0 \le i \le n-1$. Set

$$U_i = (t^{-1}k[t^{-1}] \otimes_k V) \oplus \left(\bigoplus_{s=i+1}^{n} ku_s\right).$$

Let P_i be the set of subspaces A of $\tilde{V}$ such that $A \supset t^{-1}A$, A and U_i are commensurable, and $\dim A/(A \cap U_i) = \dim U_i/(A \cap U_i)$. Define a filtration $P_i^{(0)} \subset P_i^{(1)} \subset \ldots$ of P_i by

$$P_i^{(s)} = \{A \in P_i| t^{s-1}k[t^{-1}] \otimes V \supset A \supset t^{-s+1}k[t^{-1}] \otimes V\}.$$

We regard the $P_i^{(s)}$ as projective varieties.

Consider the line bundle $\mathfrak{L}_i^* = \{(\alpha,A)| A \in P_i, \alpha \in (A|U_i)\}$ over P_i. For $s \in \mathbf{Z}_+$, we regard $\mathfrak{L}_i^*$ restricted to $P_i^{(s)}$ as an algebraic variety, and call a section of $\mathfrak{L}_i^*$ regular if its restriction to each $P_i^{(s)}$ is a regular map. Then the group $\hat{SL}(V)$ acts on the space $H^0(P_i,\mathfrak{L}_i^*)$ of regular sections of $\mathfrak{L}_i^*$.

THEOREM 4. *The* $\hat{SL}(V)$-*module* $H^0(P_i,\mathfrak{L}_i^*)^0$ *of regular sections with open stabilizer of the line bundle* $\mathfrak{L}_i^*$ *is irreducible. As* $\hat{sl}(V)$-*modules, we have* $H^0(P_i,\mathfrak{L}_i^*)^0 \cong L(\Lambda_i)$.

Note Added in Proof. Recently, Frenkel (5) independently obtained, by a different method, some of the results of section 3b of the present paper. He referred to a paper by Bardakci and Halpern (6) in which they actually construct the restriction of the spin representation of $\hat{o}(6)$ to $\hat{gl}(3)$.

We owe to G. Lusztig the idea of the construction of the group $\hat{SL}_n(k[t,t^{-1}])$, which inspired our work on the wedge representation. We would like to thank him for discussions on this matter. The research has been partially supported by National Science Foundation Grants MCS-8005813 and MCS-800286.

1. Berezin, F. A. (1966) *The Method of Second Quantization* (Academic, New York).
2. Chevalley, C. C. (1954) *The Algebraic Theory of Spinors* (Columbia Univ. Press, New York).
3. Kac, V. G. (1978) *Adv. Math* **30**, 85–136.
4. Parthasarathy, R. (1972) *Ann. Math.* **96**, 1–30.
5. Frenkel, I. B. (1980) *Proc. Natl. Acad. Sci. USA* **77**, 6303–6306.
6. Bardakci, K. & Halpern, M. B. (1971) *Phys. Rev. D* **3**, 2493–2506.