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Mathematics

The hard-hexagon model and Rogers–Ramanujan type identities

(statistical mechanics/combinatorial identities/basic hypergeometric series)

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ABSTRACT In regime II of Baxter's solution of the hard-hexagon model [Baxter, R. J. (1980) *J. Phys.* A 13, L61–L70], he presents six conjectures identifying certain one-dimensional partition functions with infinite products. An outline of the proof of these conjectures is given here.

1. Introduction

In 1980, Baxter (1) found his beautiful solution to the hard-hexagon model of statistical mechanics. His treatment of this model is naturally divided into four regimes that depend on values taken by various parameters associated with the model. Then in truly astounding fashion it turns out that eight Rogers–Ramanujan type identities [all essentially known to Rogers (2, 3)] are the fundamental keys for finding infinite product representations of the related statistical mechanics partition functions in regimes I, III, and IV. Indeed, the required identities are these:

Regime I

$$\sum_{n=0}^{\infty} q^{n^2}/(q)_n = 1/\Pi(q,q^4;q^5); \qquad [1.1]$$

$$\sum_{n=0}^{\infty} q^{n^2+n}/(q)_n = 1/\Pi(q^2,q^3;q^5); \qquad [1.2]$$

Regime III

$$\sum_{n=0}^{\infty} q^{n(3n-1)/2}/(q)_n(q;q^2)_n = \Pi(q^4,q^6,q^{10};q^{10})/(q)_\infty; \qquad [1.3]$$

$$\sum_{n=0}^{\infty} q^{3n(n+1)/2}/(q)_n(q;q^2)_{n+1} = \Pi(q^2,q^8,q^{10};q^{10})/(q)_\infty; \qquad [1.4]$$

Regime IV

$$\sum_{n=0}^{\infty} q^{n(n+1)}/(q)_{2n+1} = \Pi(q^3,q^7,q^{10};q^{10})\Pi(q^4,q^{16};q^{20})/(q)_\infty; \qquad [1.5]$$

$$\sum_{n=0}^{\infty} q^{n(n+1)}/(q)_{2n} = \Pi(q,q^9,q^{10};q^{10})\Pi(q^8,q^{12};q^{20})/(q)_\infty; \qquad [1.6]$$

$$\sum_{n=0}^{\infty} q^{n^2}/(q)_{2n} = 1/\Pi(q^4,q^{16};q^{20})(q;q^2)_\infty; \qquad [1.7]$$

$$\sum_{n=1}^{\infty} q^{n^2}/(q;q^2)_{2n-1} = q/\Pi(q^8,q^{12};q^{20})(q;q^2)_\infty. \qquad [1.8]$$

We are herein utilizing the following standard notation of Slater (4)

$$(a)_n = (a;q)_n = \prod_{j=0}^{n-1}(1-aq^j); \qquad [1.9]$$

$$(a)_\infty = (a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j); \qquad [1.10]$$

$$\Pi[a_1,a_2,\ldots,a_r;q] = (a_1)_\infty(a_2)_\infty\ldots(a_r)_\infty. \qquad [1.11]$$

These results either are given explicitly by Rogers (2, 3) or are immediate consequences of his work: Eq. 1.1 is equation 1 on p. 328 of ref. 2; Eq. 1.2 is equation 2 on p. 329 of ref. 2; Eq. 1.3 is equation 2, line 3, on p. 330 of ref. 3; Eq. 1.4 is equation 2, line 2, on p. 330 of ref. 3; Eq. 1.5 is implicit in the identity of equations 2 and 3 on p. 330 of ref. 2 when $\lambda = 1$ and q is replaced by $-q$ (explicitly given by Slater (5), equation 94); Eq. 1.6 is equation 13 on p. 332 of ref. 2; Eq. 1.7 is the second equation of p. 331 of ref. 2; Eq. 1.8 is equation 3, line 2, on page 330 of ref. 3.

For regime II, however, it turns out that one must consider the following rather complicated one-dimensional partition function:

$$F_k(\sigma_1) = \lim_{m\to\infty} \sum_{\sigma_2,\sigma_3,\ldots,\sigma_m} q^{\Sigma i(\sigma_i\sigma_{i+2}-\sigma_{i+1}+\bar{\sigma}_{i+1})}, \qquad [1.12]$$

in which the summation runs over all possible $(m-1)$-tuples $(\sigma_2,\ldots,\sigma_m)$ subject to the conditions: (*1*) $\sigma_j = 0$ or 1 for $1 \leqq j \leqq m$; (*2*) $\sigma_j + \sigma_{j+1} \leqq 1$ for $1 \leqq j \leqq m-1$; (*3*) $\sigma_m = \sigma_{m+1} = 0$; (*4*) $\bar{\sigma}_i = 1$ if $i \equiv k \pmod 3$, otherwise $\bar{\sigma}_i = 0$. Baxter obtains recurrence relations for refinements of these functions $F_k(\sigma_1)$; however, the techniques that he applies successfully to solve the recurrence relations in the other three regimes fail here. For this reason he is unable to find counterparts of the infinite series in Eqs. 1.1–1.8. By direct expansion he obtains overwhelming evidence to conjecture that each of $F_1(0)$, $F_1(1)$, $F_2(0)$, $F_2(1)$, $F_3(0)$, and $F_3(1)$ are identical with elegant combinations of infinite products.

In *Section 2* we shall give double series expansions for the $F_k(\sigma_1)$ that indeed establish all six of Baxter's conjectures. Apart from their contribution to Baxter's solution of the hard-hexagon model, these results are also surprising mathematically. They are not apparently limiting cases of known basic hypergeometric series identities; this is in contradistinction to the fact that the place of Eqs. 1.1–1.8 in the hierarchy of basic hypergeometric series is well known [cf. Slater (5), Bailey (6, 7)]. In *Section 3*, we shall describe the results and techniques required to establish these theorems.

2. The identities for Baxter's conjectures

THEOREM 1.

$$F_1(0) = \sum_{n=0}^{\infty} \sum_{0\leq r\leq(3n+1)/2} \frac{q^{(3n^2+3n)/2 - r}}{(q^2;q^2)_r(q)_{3n-2r+1}}$$

$$= \{\Pi[q^4,q^{11},q^{15};q^{15}] + q\Pi[q,q^{14},q^{15};q^{15}]\}/(q)_\infty.$$

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THEOREM 2.

$$F_1(1) = \sum_{n=0}^{\infty} \sum_{0\le r\le 3n/2} \frac{q^{(3n^2+3n)/2 - r}}{(q^2;q^2)_r(q)_{3n-2r}}$$

$$= \{\Pi[q^7,q^8,q^{15};q^{15}] - q\Pi[q^2,q^{13},q^{15};q^{15}]\}/(q)_\infty.$$

THEOREM 3.

$$F_2(0) = \sum_{n=1}^{\infty} \sum_{0\le r\le (3n-1)/2} \frac{q^{n(3n-1)/2 - r}}{(q^2;q^2)_r(q)_{3n-2r-1}}$$

$$= \Pi[q^6,q^9,q^{15};q^{15}]/(q)_\infty.$$

THEOREM 4.

$$F_2(1) = \sum_{n=0}^{\infty} \sum_{0\le r\le (3n+1)/2} \frac{q^{n(3n+5)/2 + 1 - r}}{(q^2;q^2)_r(q)_{3n-2r+1}}$$

$$= q\Pi[q^3,q^{12},q^{15};q^{15}]/(q)_\infty.$$

THEOREM 5.

$$F_3(0) = \sum_{n=0}^{\infty} \sum_{0\le r\le 3n/2} \frac{q^{n(3n+1)/2 - r}}{(q^2;q^2)_r(q)_{3n-2r}}$$

$$= \Pi[q^6,q^9,q^{15};q^{15}]/(q)_\infty.$$

THEOREM 6.

$$F_3(1) = \sum_{n=1}^{\infty} \sum_{0\le r\le (3n-1)/2} \frac{q^{n(3n+1)/2 - r}}{(q^2;q^2)_r(q)_{3n-2r-1}}$$

$$= q\Pi[q^3,q^{12},q^{15};q^{15}]/(q)_\infty.$$

Baxter had conjectured the identity of each of the $F_k(\sigma_1)$ with the corresponding infinite products given above. Two major steps are needed to prove these theorems. First, one must develop methods for the treatment of the expressions given in Eq. **1.12** so that the double series representations given above can be found. Second, a set of transformations is required to allow identification of the double series with the appropriate infinite product expression.

3. Outline of the proofs of the theorems

Our attack diverges from that of Baxter immediately. Baxter's development of series-product identities relies on the taking of the limit as m tends to ∞ in Eq. **1.12**. We instead find representations for the partition functions arising in regime III with m remaining fixed and finite. We then utilize the powerful fact that when m is finite one may effectively pass from regime III to regime II by the transformation $q \to q^{-1}$. Our solution of regime III (and consequently our proof of *Theorems 1–6*) relies on the two following polynomial identities:

$$\sum_{n,r\ge 0} q^{n(3n+1)/2} \begin{bmatrix} N-2n-2r \\ n \end{bmatrix}_q \begin{bmatrix} r+n \\ r \end{bmatrix}_{q^2} q^r = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{5\lambda^2+\lambda} \begin{bmatrix} N \\ \left[\frac{N-5\lambda}{2}\right] \end{bmatrix}_q ; \quad [3.1]$$

$$\sum_{n,r\ge 0} q^{3n(n+1)/2} \begin{bmatrix} N-2n-2r-1 \\ n \end{bmatrix}_q \begin{bmatrix} r+n \\ r \end{bmatrix}_{q^2} q^r = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{5\lambda^2-3\lambda} \begin{bmatrix} N \\ \left[\frac{N-5\lambda}{2}\right]+1 \end{bmatrix}_q ; \quad [3.2]$$

in which

$$\begin{bmatrix} N \\ M \end{bmatrix}_q = \begin{cases} \dfrac{(1-q^N)(1-q^{N-1})\ldots(1-q^{N-M+1})}{(1-q^M)(1-q^{M-1})\ldots(1-q)}, & \text{for } M \geqq 0 \\ 0, & \text{for } M < 0,\ N \geqq 0, \end{cases} \quad [3.3]$$

and $[x]$ = the largest integer not exceeding x. If in identity **3.1** we let $N \to \infty$, the first result required for regime III (namely Eq. **1.3**) is obtained. Similarly if $N \to \infty$ in identity **3.2**, we obtain Eq. **1.4**.

To obtain *Theorems 1–6*, one merely replaces N by $3N + a$ ($a = 0, \pm 1$) in identities **3.1** and **3.2**, then replaces q by q^{-1}, next multiplies by the minimal power of q necessary to produce polynomials in q, and then lets $N \to \infty$. This process produces the identity of series and products described in these theorems, and the relationship between regimes II and III that follows from the replacement of q by q^{-1} provides the identity with the various $F_k(\sigma_1)$.

4. Conclusion

Of course the immediate interest of the results described here lies in the fact that the Rogers–Ramanujan type identities for regime II of the hard-hexagon model are now rigorously established. On the other hand, there are numerous interesting long-range questions more of interest in the theory of partitions and q-series that will be explored in our complete exposition:

(*i*) Suppose the Rogers–Ramanujan partition ideal [see Andrews (ref. 8, chapter 8) for detailed discussion of partition ideals] is replaced by another classical partition ideal; what happens in regimes II, III, and IV appropriately modified?

(*ii*) The $q \to q^{-1}$ duality of regimes II and III also exists between regimes I and IV. Indeed, the relevant polynomial identities for this latter relationship are

$$\sum_{j\ge 0} q^{j^2} \begin{bmatrix} N-j \\ j \end{bmatrix}_q = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda+1)/2} \begin{bmatrix} N \\ \left[\frac{N-5\lambda}{2}\right] \end{bmatrix}_q ; \quad [4.1]$$

$$\sum_{j\ge 0} q^{j^2+j} \begin{bmatrix} N-j \\ j \end{bmatrix}_q = \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{\lambda(5\lambda-3)/2} \begin{bmatrix} N+1 \\ \left[\frac{N+1-5\lambda}{2}\right]+1 \end{bmatrix}_q . \quad [4.2]$$

These identities were completely stated in ref. 9 and have their origin in the work of Schur (10). The arguments used to obtain *Theorems 1–6* from identities **3.1** and **3.2** may now be turned on identities **4.1** and **4.2** to obtain Eqs. **1.5–1.8**, a relationship previously unnoticed. Thus a "duality theory" between various sets of identities of the Rogers–Ramanujan type deserves exploration.

(*iii*) The analytic duality described above has a corresponding manifestation in the partition-theoretic interpretations of the various identities considered. Thus the well-known combinatorial interpretations of Eqs. **1.1** and **1.2** are "dual" to the combinatorial interpretations of Eqs. **1.4–1.8** (see refs. 11–13). The possible scope of this duality will also be explored.

(*iv*) One referee has pointed out that if the signs joining the two products in *Theorems 1* and *2* are reversed, one obtains expressions that can be converted to products by using the quintuple product identity (equation 7.4.7 on p. 205 of ref. 4. For example,

$$\Pi[q^4,q^{11},q^{15};q^{15}] - q\Pi[q,q^{14},q^{15};q^{15}] = \Pi[q,-q^2,-q^3,q^5,-q^7,-q^8,q^9,q^{10};q^{10}]. \quad [4.3]$$

This suggests taking the general quintuple product identity, say in the form

$$\prod_{n\geq 1}(1-a^3x^{3n-2})(1-a^{-3}x^{3n-1})(1-x^{3n})$$

$$+ a\prod_{n\geq 1}(1-a^3x^{3n-1})(1-a^{-3}x^{3n-2})(1-x^{3n})$$

$$= \prod_{n\geq 1}(1-a^2x^{2n-1})(1-a^{-2}x^{2n-1})\cdot$$

$$(1+ax^{n-1})(1+a^{-1}x^n)(1-x^n),$$

and studying the function that arises by changing the + sign on the left to a − sign.

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1. Baxter, R. J. (1980). *J. Phys. A* **13**, 161–170.
2. Rogers, J. J. (1894) *Proc. Lond. Math. Soc. First Ser.* **25**, 318–343.
3. Rogers, L. J. (1917) *Proc. Lond. Math. Soc. Second Ser.* **16**, 315–336.
4. Slater, L. J. (1966) *Generalized Hypergeometric Functions* (Cambridge Univ., Cambridge, England).
5. Slater, L. J. (1952) *Proc. Lond. Math. Soc. Second Ser.* **54**, 147–167.
6. Bailey, W. N. (1947) *Proc. Lond. Math. Soc. Second Ser.* **49**, 421–435.
7. Bailey, W. N. (1949) *Proc. Lond. Math. Soc. Second Ser.* **50**, 1–10.
8. Andrews, G. E. (1976) *The Theory of Partitions*, Encyclopedia of Mathematics and Its Applications (Addison-Wesley, Reading, MA), Vol. 2.
9. Andrews, G. E. (1970) *Scr. Math.* **28**, 297–305.
10. Schur, I. (1917) *Sitzungsber. Akad. Wiss., Berl. Kl., Math. Phys. Tech.* 302–321.
11. Gordon, B. (1965) *Duke Math. J.* **31**, 741–748.
12. Andrews, T. E. & Askey, R. (1977) in *Higher Combinatorics*, ed. Aigner, M. (Reidel, Dordrecht, The Netherlands), pp. 3–25.
13. Hirschhorn, M. D. (1979) *J. Combinatorial Theory Ser. A* **27**, 33–37.