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NEW PROPERTIES OF ALL REAL FUNCTIONS

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In a former paper[1] the author communicated a number of properties of every real function $f(x)$, which were stated in terms of the successive saltus functions associated with a given function. The present results do not involve these saltus functions and are direct qualifications of $f(x)$. Since $f(x)$ is entirely unrestricted, except, of course, that it is defined—even this restriction may be partially dispensed with—and therefore finite for every real x, these qualifications are consequences of nothing else than that $f(x)$ is a function. A new light is thus thrown upon the nature of a function.

The new properties are of two kinds, descriptive and metric; the former are concerned with density and the latter with measure (Lebesgue).

For the sake of greater concreteness, we discuss, for the most part, planar sets and real, single-valued functions of two real variables.

1. *Descriptive Properties.*—We say that a planar set S is an "I-region" (= open set) if every point of S is an inner point of S. We deal with binary relations $\mathfrak{R}$ between I-regions and points. $I\mathfrak{R}P$ shall mean that the I-region I has the relation $\mathfrak{R}$ to the point P. The relation $\mathfrak{R}$ is said to be "closed" if the relationships $I\,\mathfrak{R}\,P_n$ and $\lim_{n \to \infty} P_n = P$ imply $I\,\mathfrak{R}\,P$. By a "neighborhood" of a point P, we understand an I-region containing P; by a "partial neighborhood" of P, an I-region of which P is an inner or a boundary point.

We have the following

Lemma I. If $\mathfrak{R}$ is a closed relation, then the points for which (*a*) $N\mathfrak{R}P$ *for every neighborhood N of P, and* (*b*) *a partial neighborhood $N_<$ exists such that* $\overline{N_< \,\mathfrak{R}\, P}$ (i.e., $N_< \,\mathfrak{R}\, P$ is false) *constitute a non-dense* (i.e., nowhere dense) *set.*

An important example of a closed relation appears in connection with a function. Let $z = f(x, y)$ be any given one-valued function of the two real variables x and y, and let f be defined for the entire XY plane. Let $\mathfrak{R} = \mathfrak{R}_{r_1 r_2}$, where $r_1 < r_2$ are two real numbers, be defined as follows: $I \mathfrak{R}_{r_1 r_2} P$ if and only if an infinite sequence of points P_n of the I-region I exists such that $\lim_{n \to \infty} P_n = P$, $\lim_{n \to \infty} f(P_n)$ exists, and $r_1 \leqq \lim_{n \to \infty} f(P_n) \leqq r_2$; here P, $P_n = (x, y)$, (x_n, y_n) and $f(P), f(P_n) = f(x, y), f(x_n, y_n)$.

Definition. The function $f(x, y)$ is said to be "densely approached at the point (ξ, η)" or in other words, the point $(\xi, \eta, f(\xi, \eta))$ of the "surface" $z = f(x, y)$ is said to be densely approached, if for every positive ϵ there exists a planar neighborhood N of (ξ, η) such that the points of N for which $|f(x, y) - f(\xi, \eta)| < \epsilon$ form a dense set in N.

Definition. An "exhaustible" set (= set of first category, according to Baire) is the sum of $\aleph_0$ non-dense sets; a "residual" set, the complement (with respect to the XY plane) of an exhaustible set.[2]

By the aid of Lemma I and the closed relation $\mathfrak{R}_{r_1 r_2}$, we prove

Theorem I. For every real function $f(x, y)$ whatsoever, the points of the surface $z = f(x, y)$ that are densely approached form a residual set. Conversely, given any residual set R whatsoever, a function $f(x, y)$ exists that is densely approached at and only at the points of R.

The following definition of dense approach is equivalent: The function $f(x, y)$ is said to be densely approached at $P = (\xi, \eta)$ if, for every partial neighborhood $N_<$ of P, the set of points of the surface $z = f(x, y)$ corresponding to the points of $N_<$ has $(\xi, \eta, f(\xi, \eta))$ as a limit point. By the use of this definition, we get the following theorem, which is equivalent to Theorem I, but which shows better, perhaps, the remarkable degree of "microscopic symmetry" an unconditioned function possesses.

Theorem I'. With every function $f(x, y)$ whatsoever, there is associated a residual set R—dependent on f—of the XY plane such that if $P = (\xi, \eta)$ is a point of R and $N_<$, a partial neighborhood of P, then $(\xi, \eta, f(\xi, \eta))$ is a limit point of the set of points $(x, y, f(x, y))$ for which (x, y) is in $N_<$.

Definition. The function f is said to be "inexhaustibly approached" at the point P if every neighborhood of P contains, for every $\epsilon > 0$, an inexhaustible set of points—i.e., a set that is not exhaustible—at which f differs from $f(P)$ by less than ϵ.

If M is any planar set, we use, in connection with approach, the expression "via M" to designate that (x, y) is restricted to range in M. Thus "f is inexhaustibly approached at P via M" means that for every neighborhood N of P and every $\epsilon > 0$, the set MN, which is the aggregate of points common to M and N, contains an inexhaustible set of points at which f differs from $f(P)$ by less than ϵ.

We have the following

Theorem II. For every function $f(x, y)$, there exists in the XY plane a *residual set R—dependent on f—such that if P is a point of R, and* $N_<$, *a partial neighborhood of P, the function f is inexhaustibly, and therefore densely approached at P via* $RN_<$.

With the aid of Theorem II we prove

Theorem III. With every function f(x, y) there is associated—not uniquely, however—a dense set D of the XY plane, such that f is continuous if (x, y) ranges over D.

2. *Generalizations.*—The considerations and theorems of Section 1 apply not only to functions of n real variables, but to every space S that satisfies the following four conditions:

(1) S is metric;[3] that is to say, with every pair of elements P and Q of S there is associated a non-negative, real number $\overline{PQ}$ (Fréchet's écart) in such a way that if P, Q and R are three elements of S, then

(*a*) $\overline{PQ} = \overline{QP}$;

(*b*) $\overline{PQ} = 0$, when and only when $P = Q$; and

(*c*) $\overline{PQ} + \overline{QR} \geqq \overline{PQ}$.

(2) S is complete (vollständig);[4] that is to say, if $\{P_1, P_2, \ldots, P_n \ldots\}$ is a "regular" sequence of elements of S—in other words, for every $\epsilon > 0$ there exists an integer n_ϵ such that $\overline{P_\lambda P_\mu} < \epsilon$ for $\lambda > n_\epsilon$ and $\mu > n_\epsilon$—there exists a limit element P (i.e., an element P with the property $\lim_{n \to \infty} \overline{P_n P} = 0$).

(3) S contains a denumerable subset that is dense in S.

(4) S has no isolated points.

We may thus state the following theorem—the definitions of the terms for S may be obtained after slight and evident modifications from those for the plane.

Theorem IV. Let S be any complete, metric space containing a dense, denumerable subset and without isolated points; and f(P), any real function defined for the elements P of S. Then there exists a residual set R, such that if P is a point of R, and $N_<$ *a partial neighborhood of P, the function f is inexhaustibly and therefore densely approached at P via* $RN_<$. *Also there exists a dense subset D of S such that f (P) is continuous if P ranges over D.*

As particular examples of a complete, metric space, with a dense, denumerable subset and without isolated points, we mention:

(*a*) Euclidean n-space where the écart between two points is the euclidean distance between them.

(*b*) A perfect subset of euclidean space.

(*c*) Hilbert space, that is, the ensemble of sequences $(x_1, x_2, \ldots x_n, \ldots)$ of real numbers with convergent $\sum_{n=1}^{n=\infty} x_n^2$. The écart between two "points" $(x_1, x_2, \ldots x_n, \ldots)$ and $(y_1, y_2, \ldots y_2, \ldots)$ is defined to be $\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \ldots}$.

(*d*) Function space: S consists of all real, continuous functions $f(x)$

defined for $0 \leqq x \leqq 1$. The écart between $f_1(x)$ and $f_2(x)$ is defined to be $\max|f_1(x) - f_2(x)|$.

The assumption that f is single-valued may also be dropped without invalidating Theorems I and II—Theorem III, of course implies single-valuedness by its very nature. We thus get the following generalization of Theorem I (and a similar one for Theorem II).

Theorem I $^{(a)}$. *Let $f(x, y)$ be any real function defined for the entire XY plane and taking at every point at least one value—the number of values may change, however, from point to point and vary from 1 to c, the cardinal number of the continuum. Then the points (x, y) such that every surface point $(x, y, f(x, y))$ is densely approached by the surface $z = f(x, y)$ constitute a residual set.*

3. *Metric Properties.*—As in the case of Section 1, we discuss in this section planar sets and functions of two variables.

Let S be any planar set; P, a point of S; C_r, a circle with P as center and r as radius; $m(C_r)$, the area of C_r, and $m_e(SC_r)$, the exterior Lebesgue measure of the portion of S in C_r. Then if

$$\lim_{r \to 0} \frac{m_e(SC_r)}{m(C_r)}$$

exists and is equal to k, we say that the "exterior metric density" of S at the point P is k. We have the following

Theorem V.[5] *Let S be any planar set. Then the points of S at which the exterior metric density of S is* $\neq 1$—i.e., the points where the exterior metric density either does not exist or does exist and is less than 1—*constitute a set of zero measure (Lebesgue).*

Definition. $N_<$ is said to be a "*non-vanishing* partial neighborhood of P," if the exterior metric density of $N_<$ at P is $\neq 0$. We have the following lemma, which corresponds to Lemma I for the descriptive properties:

Lemma II. Let $\Re$ be a closed relation as in Lemma I. The points P for which (a) $N \Re P$ for every neighborhood of P and (b) a non-vanishing partial neighborhood $N_<$ exists such that $\overline{N_< \Re P}$ (i.e., *$N_< \Re P$ is false) constitute a set of zero measure.*

By the aid of this lemma we prove

Theorem VI. Let $f(x, y)$ be any real, one-valued function defined in the entire plane. Then there exists in the XY plane a set Z—dependent on f—of measure zero, such that if (a) (x, y) is any point of the XY plane not belonging to Z; (b) $N_<$, any non-vanishing partial neighborhood of (x, y); and (c) S, any sphere with $(x, y, f(x, y))$ as center; then there is at least one point of the surface $z = f(x, y)$ lying in the sphere S and having as projection upon the XY plane a point in $N_<$.

This theorem becomes false if we omit the restriction that the partial neighborhood $N_<$ shall be non-vanishing.

By means of Lemma II, we prove also

Theorem VII. The set of points P at which f is inexhaustibly approached and for which a non-vanishing partial neighborhood exists via which f is exhaustibly approached constitute a set of measure zero.

Definition. f is said to be "neglectably approached at the point $P \equiv (\xi,\eta)$ via M" if a sphere S exists with $(\xi, \eta, f(\xi, \eta))$ as center, such that the projection upon the XY plane of the set of points common to S and the surface $z = f(x, y)$ has a set of measure zero in common with M.

The following theorem generalizes Theorem VI.

Theorem VIII. Let $z = f(x, y)$ be any real, one-valued function defined in the XY plane. Then the points P of the XY plane that possess a non-vanishing partial neighborhood via which f is neglectably approached at P constitute a set of zero measure.

Definition. f is "quasi-continuous"[6] at P if for every ϵ the set of points Q for which $|f(Q) - f(P)| < \epsilon$ has 1 as exterior metric density at P.

We have the following theorem, which generalizes theorem *VIII.*

Theorem IX. f is quasi-continuous except at the points of a set of measure zero.

Concluding Remarks.—As in the case of the descriptive properties, the metric theorems may be extended to many-valued functions. Theorem VIII, for example, when thus generalized reads as follows: *Let $z = f(x, y)$ be any real, single- or many-valued function defined in the entire XY plane. Then the points (x, y) of the XY plane for which a surface point $(x, y, f(x, y))$ and a non-vanishing partial neighborhood $N_<$ exist such that $(x, y, f(x, y))$ is neglectably approached via $N_<$ constitute a set of zero measure.*

The metric properties hold for functions of a single variable and, in general, for functions of n variables. Extension to $\aleph_0$ space, to function space and to more general spaces would require a satisfactory definition of measure for such spaces;[7] it is not our purpose in this paper to enter upon such questions.

Instead of projecting the surface points of $z = f(x, y)$ upon the XY plane, we may project them upon the X-axis and thus obtain other properties. For example, let us define the relationship $\Re_{r_1r_2r_3r_4}$—between I-regions and points of the X-axis—as follows: $I \; \Re_{r_1r_2r_3r_4} \; \xi$ if the surface points having x-coördinates in I have a limit point in the rectangle $x = \xi$ $r_1 \leqq y \leqq r_2$, $r_3 \leqq z \leqq r_4$. $\Re_{r_1r_2r_3r_4}$ is closed. By applying Lemma II to this closed relation, we obtain the following result: *Let $z = f(x, y)$ be any single- or many-valued function defined in the entire XY plane. Let ξ be a point of the X-axis of the following character: a surface point (ξ, η, ζ) and a partial non-vanishing (linear) neighborhood $N_<$ of ξ exist such that (ξ, η, ζ) is not a limit point of surface points having x-coördinates in $N_<$. The totality of points ξ is of measure zero.*

Similar results may be obtained for other metric properties and also in the case of the descriptive properties. In the case of a function of n-variables, we may project upon an $(n - 1)$ space, an $(n - 2)$ space, etc.

The proofs of the preceding theorems are contained in a paper that has been offered to the *Transactions of the American Mathematical Society.*

[1] *Ann. Math. Princeton*, **18**, 1917 (147).

[2] For the terminology cf. Denjoy, *J. Math., Paris*, ser. 7, **1**, 1915 (122–125).

[3] Cf. for example, Fréchet, *Rend. Circ. Math. Palermo*, **22**, 1907, p. 1; and Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 211.

[4] Hausdorff, l. c., p. 315.

[5] For the case where only measurable sets are admitted cf., for example, de la Vallée Poussin, *Cours d' Analyse*, **2**, 1912, p. 114. For the linear case of general (not necessarily measurable) sets cf. Blumberg, *Bull. Amer. Math. Soc.*, **25**, 1919 (350).

[6] Cf. Denjoy, *Bull. Soc. Math. France*, **43**, 1915 (165).

[7] In this connection cf. Gâteaux, *Ibid.*, **47**, 1919 (47).

GENERALIZED LIMITS IN GENERAL ANALYSIS

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It is well known that to each of the various methods for summing divergent series there corresponds an analogous method for summing divergent integrals. It is readily seen that similar methods may be used for obtaining types of generalized derivatives of a function at a point where the ordinary derivative fails to exist. Likewise, in any other case in Analysis where we wish to associate a limit with a variable that oscillates, we will naturally be led to make use of methods that have been tried out in the case of divergent series.

It would be manifestly poor economy of time and thought to elaborate for each of these special theories such fundamental results as are common to them all, if these results can be obtained in one central theory that includes all the others. According to a principle of generalization formulated by E. H. Moore and stated by him on several occasions,[1] the existence of such a general theory is implied by the analogies found among the various special theories. It is natural to designate this general theory as the theory of generalized limits in General Analysis.

It is the purpose of the present communication to illustrate the nature and usefulness of this theory by outlining the proof of a theorem in it which is a generalization of one of the important theorems in the theory of divergent series. This latter theorem is the Knopp-Schnee-Ford theorem[2] with regard to the equivalence of the Cesàro and Hölder means of order k