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Mathematics

Post-classification version of Jordan's theorem on finite linear groups

(finite simple groups/groups of Lie type/algebraic groups)

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ABSTRACT **Using classification of finite simple groups, I show that a finite subgroup G of $GL_n(C)$, where C = the complex numbers, contains a commutative normal subgroup M of index at most $(n + 1)!n^{a\log n+b}$. Moreover, if G is primitive and does not contain normal subgroups that are direct products of large alternating groups, then the factor $(n + 1)!$ can be dropped. I further show that similar statements hold also in characteristics $p \geqq 2$, if one takes M to be an extension of a group of Lie type of characteristic p by a solvable group that has a normal p-subgroup with commutative p'-quotient. These results improve the celebrated theorems of Jordan and of Brauer and Feit.**

1. C. Jordan's theorem says that there exists a function $f : N \to N$ (from natural numbers to natural numbers) such that any finite subgroup of $GL_n(C)$ contains a normal commutative subgroup of index $\leqq f(n)$.

R. Brauer and W. Feit (1) generalized Jordan's theorem to fields k of characteristic $p \geqq 2$. They constructed a function $F : N \times N \times N \to N$ such that any finite subgroup H of $GL_n(k)$ contains a commutative normal p'-subgroup A of index $\geqq F(p, n, r)$ if $p^{r+1} \nmid H$. It is classically known that dependence on p and r cannot be excluded: the tower of groups $GL_n(F_{p^i})$, in which $i = 1, 2, \ldots$, furnishes an example. M. Nori (personal communication) recently discovered a result that can be considered as a conceptual refinement of a particular case of Brauer–Feit theorem; both his result and methods apply only to subgroups of $GL_n(F_p)$ with p large and depending on n. The present work was inspired by this result of Nori. I remove the above-mentioned restrictions (by a different method) and show that only normal p-subgroups and finite simple groups of Lie type and of characteristic p are responsible for the difference between characteristic 0 and characteristic $p \geqq 2$.

2. For a field k, let $p = p(k)$ denote its characteristic exponent ($p = 1$ if char $k = 0$ and p = char k otherwise). A 1-group and a group of characteristic 1 are both trivial, whereas a 1′-group is an arbitrary group. I use "group of Lie p-type" in place of "group of Lie type and of characteristic p." For a group M I denote Aut M and Out M, respectively, as the automorphism group and the group of outer automorphisms of M and use $Z_M(R)$ for the centralizer and $N_M(R)$ for the normalizer in M of a subset R of M. For p, a prime $O_p(M)$ denotes the largest normal p-torsion subgroup of M; we say that M is a p'/p-group if $M/O_p(M)$ is a commutative p'-group. The symmetric and alternating groups on n letters are denoted Sym_n and Alt_n; log denotes $\log_2$.

3. The main result is *Theorem 1*.

THEOREM 1. *For every field* k *with* p = p(k) *and every finite subgroup* H *of* $GL_n(k)$ *there exist*
(i) *a normal* p'/p*-subgroup* T *of* H;
(ii) *a normal subgroup* B *of* H, $B \supseteq O_p(H)$, *with* $B/O_p(H)$ *isomorphic to a central extension of a direct product of groups* Alt_{m_i}, $i = 1, \ldots, t$, $m_i \geqq 8$, $2^t \Pi_{1\leq i\leq t} [(m_i - 2)/3] \leqq n$;
(iii) *a normal subgroup* L *of* H, $L \supseteq O_p(H)$, *with* $L/O_p(H)$ *isomorphic to a central extension of a direct product of finite simple groups* L_i *of Lie* p*-type*, $i = 1, \ldots, m$, *with* $m \leqq \log(2^{-t}n/\Pi_{1\leq i\leq t} [(m_i - 2)/3])$; *and*
(iv) *a homomorphism* $\phi : H \to Sym_n$, *such that*
(a) $\ker \phi \subseteq TBL$,
(b) $|H/TL| \leqq g(n) [(3n + 9)/2]!$, *and*
(c) *if* H *is primitive, then* $|\ker \phi/TBL| \leqq g(n)$, *where* $g(n) = n^{8\alpha^2 \log n+b}$, $\alpha = (\log 3 - 1)^{-1}$, *and* b *is an appropriate constant*.

Remark. (*i*) This result implies one of R. Brauer and W. Feit with $F(p, n, r) = p^{7r} g(n) [(3n + 9)/2]!$. H. Bass' generalization (2) of Jordan's theorem can be formulated also similarly to *Theorem 1*. (*ii*) It should be possible to replace $[(3n + 9)/2]$ in *Theorem 1* part *b* by $(n + 2)$. (*iii*) A better bound is known for solvable groups (see ref. 3).

In characteristic 0, these methods give a better result.

THEOREM 2. *Let* H *be a finite subgroup of* $GL_n(C)$. *Then there exist*
(i) *a normal commutative subgroup* A *of* H;
(ii) *a normal subgroup* B *of* H *isomorphic to a direct product of groups* Alt_{m_i}, $i = 1, \ldots, t$, $m_i \geqq 8$, $\Pi_{1\leq i\leq t} (m_i - 1) \leqq n$; *and*
(iii) *a homorphism* $\phi : H \to Sym_n$ *such that*
(a) $\ker \phi \supseteq AB$,
(b) $|H/A| \leqq g_0(n)(n + 1)!$, *and*
(c) *if* H *is primitive then* $|\ker \phi/AB| \leqq g_0(n)$ *where* $g_0(n) = g(n) \cdot n^{b'}$ *with an appropriate constant* b′.

Theorems 1 and *2* can be further sharpened if one introduces additional parameters into the statements or in the estimates. The proof can be followed through for any particular n, with the result being a description of finite linear groups of degree n; there is a large literature on this subject (see section 13 of ref. 4). In a different direction my results can be viewed as generalizations of some results of R. Brauer, W. Feit, J. H. Lindsey, D. A. Sibley, and others (see, e.g., section 10 of ref. 4).

Further, using *Theorem 2* (or the ideas used in its proof), one can obtain that there is an $n(r)$ such that the subgroup $(Sym_{n+1})^r \rtimes Sym_r$ of $PGL_{n^r}(C)$ [with $(Sym_{n+1})^r$ embedded via $\phi^{\otimes r}$, where ϕ is the faithful irreducible representation of Sym_{n+1} of degree n and where Sym_r permutes the factors] is a maximal finite subgroup of $PGL_{n^r}(C)$ if $n \geqq n(r)$. There are, clearly, other statements of the same kind that also follow from *Theorem 2*. In this connection, see ref. 4, the end of section 14, and, for $r = 1$, see ref. 5.

4. In the proof of *Theorems 1* and *2* we use *Propositions 1*, *2*, and *3*.

PROPOSITION 1. *There exists a finite list* Lst *of isomorphism classes of finite simple groups of Lie type such that for any* n, *every finite set* $p_1, \ldots, p_r$ *of primes, every set* L_i, i = 1, …, r, *of finite simple groups of Lie* p_i*-type that are not isomorphic to groups from* Lst, *and a faithful irreducible*

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projective representation of $L_1 \times \ldots \times L_r$ *of degree* $\leqq n$ *over a field* k, *the following hold*:

(i) *If* $p_i \neq p(k)$, $i = 1, \ldots, r$, *then*

(a) $|L_1 \times \ldots \times L_r| \leqq n^{8\alpha^2 \log n + 11 + 2\alpha}$ *and*

(b) *Out* $(L_1 \times \ldots \times L_r)$ *is isomorphic to a subgroup of* Sym_n; *and*

(ii) *if* $p_i = p(k)$, $i = 1, \ldots, r$, *then* $N_{GL_n(k)}(\Pi L_i)/Z_{GL_n(k)}(\Pi L_i)\cdot \Pi L_i$ *is isomorphic to a subgroup of* Sym_n.

PROPOSITION 2. *Let* k *be a field. If* $Alt_{m_1} \times \ldots \times Alt_{m_t}$, $m_i \geqq 8$, $i = 1, \ldots, t$ *has over* k *a faithful irreducible projective representation of degree* n, *then*

$$|Aut(Alt_{m_1} \times \ldots \times Alt_{m_t})| \leqq [(3n + 9)/2]!$$

PROPOSITION 3 (see ref. 6, section 44). *If* Alt_m, $m \geqq 8$, *has a faithful irreducible projective representation of degree* n *over* **C**, *then* $m \leqq n + 1$. *Moreover, if this representation does not lift to a linear representation of* Alt_m, *then* $m \leqq 1 + 2 \log n$.

5. I now describe how *Theorems 1* and *2* follow from *Propositions 1–3*. Let H be a finite subgroup of $GL_n(k)$, where k is a field that can be assumed to be algebraically closed. Write V for k^n. Let $V_0 = V \supset V_1 \ldots \supset V_s = \{0\}$ be a Jordan–Hölder sequence for the H-module V. We have homomorphisms $h_i : H \to GL(V_{i-1}/V_i)$, $i = 1, \ldots, s$. Then $\oplus h_i : H \to \Pi GL(V_{i-1}/V_i)$. We have $\ker(\oplus h_i) = O_p(H)$. Therefore, if we prove *Theorems 1* and *2* for $(\oplus h_i)(H)$, the general form will follow immediately. Set $V_i' = V_{i-1}/V_i$, $i = 1, \ldots$. Let T_i, B_i, $L_{(i)}$, and ϕ_i be the objects claimed by *Theorem 1* for $h_i(H) \subseteq GL(V_i')$. Set $T = \cap h^{-1}(T_i)$, $B = \cap h^{-1}(B_i)$, $L = \cap h^{-1}(L_{(i)})$, and $\phi = \oplus \phi_i$. Then T, B, L, and ϕ have properties *i–iv* and *a* of *Theorem 1*. Set $G_a(x) = \Gamma((3x + 11)/2)\cdot x^{8\alpha^2 \log x + a}$. In view of the dominant growth of the Γ-function, for every $a \in \mathbf{R}$, $a > 1$, there exists a $c = c(a) \in \mathbf{R}$ such that $\Pi G_a(x_i) \leqq G_{a+c}(x_i)$ if $x_i \geqq 2$. Thus, if *Theorem 1* holds for irreducible V, it holds for all V (with an appropriately larger b). Suppose that H is imprimitive. Let $V = \oplus_{1 \leq i \leq r} V_i''$ be a minimal imprimitivity system for H on V. This gives us a homomorphism $\omega : H \to Sym_r$. Let W_i, $i = 1, \ldots, r$, be an irreducible ker ω-submodule of V_i''. An argument similar to one we used to pass to irreducible modules shows that it is enough to consider the restriction of ker ω to W_1.

Thus, we can assume that H is primitive. Take then T to be the center of H. Let S be the socle of H/T. Let $\overline{S}_{ab}$ and $\overline{S}_{nab}$ be its commutative factor and its complement. Let S_{ab} and S_{nab} be their preimages in H. One easily sees (as in ref. 7, for example) that S_{ab} is a central product of an extra-special group and T. Then $S_{ab} = d^2$ for some integer d and the S_{ab}-module V is a multiple of a simple d-dimensional S_{ab}-module. Write $d = p_1^{a_1} \ldots p_r^{a_r}$ where the p_i are different primes. The action of H on itself by conjugation defines a homomorphism $\omega' : H \to Aut_T S_{ab}$, the group of automorphisms trivial on T; $Aut_T S_{ab}$ is a subgroup of $\overline{S}_{ab} \rtimes \Pi_{1 \leq i \leq r} Sp_{2a_i}(\mathbf{F}_{p_i})$. Thus, $|H/\ker \omega'| \leqq |Aut_T S_{ab}| \leqq d^2\, \Pi_{1 \leq i \leq r}\, p_i^{2a_i^2 + a_i} \leqq d^2\, (\Pi p^{a_i})\, (\Pi p^{2a_i})^{\max a_i} \leqq d^{3 + 2\log d}$. We turn to S_{nab} now. Let $\overline{S}_1, \ldots, \overline{S}_u$ be the simple components of $\overline{S}_{nab}$. Assume that $\overline{S}_i \simeq Alt_{m_i}$, $m_i \geqq 8$, $i = 1, \ldots, u_1$, that $\overline{S}_i$, $i = u_1 + 1, \ldots, u_2$ are simple finite groups of Lie p'-type, and that $\overline{S}_i$, $i = u_3 + 1, \ldots, u$ are sporadic finite simple groups or groups Alt_h, $5 \leqq h \leqq 7$ or groups from *Lst*. One easily sees that the orders of Aut $\overline{S}_i$, $i = u_3 + 1, \ldots, u$ are bounded by v, where v is the maximum of the orders of the automorphism groups of the sporadic simple groups. Therefore $|\Pi_{u_3 \leq i \leq u}$ Aut $S_i| \leqq v^{\log n} = n^{\log v}$.

The action of H on itself by conjugation determines a homomorphism $\omega'' : H \to$ Aut S_{nab} and, by composition, a homomorphism $\overline{\omega} : H \to$ Out S_{nab}. We clearly have ker $\omega' \cap$ ker $\omega'' = T$. Now *Theorem 1* in the primitive case follows from the above discussion and *Propositions 1* and *2* with $b = 11 + 2\alpha + \log v$.

Theorem 2 follows from the similar argument. We only have to separate among $\overline{S}_1, \ldots, \overline{S}_{u_1}$ those whose preimage in H is isomorphic to $\overline{S}_i \times T$. For the remaining ones, we have a logarithmic estimate from *Proposition 3*.

6. I now outline proofs of *Propositions 1* and *2*. Let L be a finite simple group of Lie p-type over a field $\mathbf{F}_q$, where q is a power of p, let d be the dimension of the corresponding algebraic group G, and let c be the order of the schematic center of G. By inspection of the table on p. 419 of ref. 8, we see that but for a finite number of exceptions which we include in *Lst*, the degrees of faithful irreducible projective representations of L are bounded from below by $(q^b - 1)/2$ for an appropriate b. It is known (or can be obtained by inspection of orders of such groups) that $|L| \leqq q^d$. Next one has $d \leqq 2b^2 + b + 11$. Therefore, in conditions of *Proposition 1*, $|L_1 \times \ldots \times L_r| \leqq \Pi q_i^{2b_i^2 + b_i + 11}$ and $2^{-r}\Pi(q_i^{b_i} - 1) \leqq n$. The latter inequality implies that $\Pi q_i^{b_i} \leqq n^2$. As in the proof of the estimate for S_{ab}, this gives $|L| \leqq n^{8\alpha^2 \log n + 2\alpha + 11}$. This is the claim of *Proposition 1* part *ia*. Now *Proposition 1* part *ib* follows from the above estimates and the description in ref. 9 of the automorphism groups of finite simple groups of Lie p-type. Part *ii* of *Proposition 1* also follows from the description of automorphism groups together with an observation that presence in $N_{GL_n(k)}(L_i)/Z_{GL_n(k)}(L_i)$ of graph and field automorphisms imposes strong restrictions on the highest weight of the corresponding representation of the algebraic group associated to L_i. This permits one to improve an estimate from below on the contribution to n from the irreducible component of L_i; then it is easy to see that Sym_n has enough room to contain $N_{GL_n(k)}(\Pi L_i)/Z_{GL_n(k)}(\Pi L_i)\cdot \Pi L_i$.

To prove *Proposition 2*, we take a prime $p \neq 2$ and consider the subgroup R in Alt_h generated by the $[(h - 2)/p]$ cycles $(1, \ldots, p)$, $(p + 1, \ldots, 2p)$, ... Since the Schur multiplier of Alt_h, $h \geqq 8$, is 2 (see ref. 6), R lifts to a subgroup $\widetilde{R}$ isomorphic to R, of the universal cover $\widetilde{Alt}_h$ of Alt_h. Then $N_{\widetilde{Alt}_h}(\widetilde{R})$ acts on $\widetilde{R}$ in the same way as $N_{Sym_h}(R)$ acts on R. Using the Clifford theory, we see that dimension of a faithful irreducible projective representation of Alt_h is not smaller than the length of the shortest nontrivial orbit of $N_{Sym_h}(R)$ on R; one easily sees that this length is $(p - 1)[(h - 2)/p]$. Thus, if $p \neq$ char k, we have an estimate $\geqq (p - 1)[(h - 2)/p]$ on dimensions of faithful irreducible projective representation of Alt_h over k. In general, one has to look over different primes. It turns out that the case $p = 3$ can always be taken for the estimate. This gives us *Proposition 2* in the case $t = 1$. Extension to the general case is straightforward; the estimates still hold because of very fast growth of factorials (or Γ-functions).

Note Added in Proof. I can now prove analogs of *Theorems 1* and *2* with $B/O_p(H)$, a direct product of alternating groups, and with $n + 2$ (resp 2, resp 60) instead of $[(3n + 9)/2]$ (resp $8\alpha^2$, resp b). The details are, however, much more involved.

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