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Escape from strange repellers

(dynamical system/mapping/cycles/derivative matrix)

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ABSTRACT In a dynamical system described by a map, it may be that a "strange" sets of points is left invariant under the mapping. The set is a repeller if points placed in its neighborhood move away. An escape rate is defined to describe this motion. An alternative method of evaluating the escape rate, based on the consideration of repulsive cycles, is proposed. In the several cases examined numerically and analytically, the escape rate is shown to agree with the proposed formula.

1. Introduction

The description of dynamical systems often involves the consideration of sets of points that are left unchanged by the flow. When these invariant sets have a complex topological structure, they are termed "strange." Both strange attractors and strange repellers play a major role in our description of dynamical systems. Attractors are important because, as the system advances, the motion can approach the attractor more and more closely.

Conversely, of course, the motion of the system tends to move away from repellers. Nonetheless a repeller might be important because, for example, it might describe a seperatrix that serves to divide two different attractors or two different types of motion. Alternatively, the motion might be one in which almost all initial points lead to an orbit that escapes to infinity. The remaining nonescaping points will then be a repeller, which might be sufficiently complex to term "strange."

One can introduce a wide variety of numbers that characterize these strange sets and the motion on them. Many of these characterizers have the nature of one kind or another of fractal dimension (1–6). In this paper, we describe the motion in the immediate vicinity of the set by an "escape rate" that states quantitatively how fast the repulsion occurs.

To define the escape rate, imagine that we have a mapping f that maps a point $\mathbf{r}$ in a manifold M into another point in the manifold. Consider some finite-sized region R within the manifold. Unlike the repeller, R is a set that is supposed to be very simple and not strange in any way whatsoever. Start with a set of N_0 initial points that are distributed uniformly (with Lebesque measure) within R. Let the mapping f operate n times and find out how many of the initial points lie in R after n iterations. Call the number N_n. As N_0 goes to infinity and n remains fixed, the staying ratio $\Gamma_n = N_n/N_0$ will approach a limit (6, 7). The escape rate α is then defined via

$$\alpha = -\lim_{n\to\infty} \frac{\ln \Gamma_n}{n}, \qquad [1.1]$$

so that exponential decrease of the number of points in R implies nonzero α. If R contains an attractor, α will be zero; if it contains a repeller but no attractor, α may be a finite positive number; if R contains neither repeller nor attractor, one may expect α to be infinite.

The quantity Γ_n is then of considerable physical interest, but it is hard to calculate, especially if M has a high dimensionality. There is an alternative approach based on the set of repulsive cycles of f, which gives a related quantity that is much easier to calculate. Let $\mathbf{r}$ be an element of the set Fix f^n, if $\mathbf{r} = f^n(\mathbf{r})$ and if this fixed point of f^n is repulsive. Then define

$$A_n = \sum_{\mathbf{r}\in \mathrm{Fix}\, f^n} \frac{1}{|\det[1 - \mathrm{D}f^n(\mathbf{r})]|}. \qquad [1.2]$$

Here, if the manifold is of dimension d, 1 is a d by d unit matrix and $\mathrm{D}f^n$ is the derivative matrix. The basic idea we propose is that, for large n, A_n and Γ_n are proportional to one another. In particular we define an exponential decay rate for A_n in analogy to Eq. **1.1** as

$$\delta = -\lim_{n\to\infty} \frac{\ln A_n}{n}. \qquad [1.3]$$

We then assert the basic identity that, for a wide variety of maps,

$$\delta = \alpha. \qquad [1.4]$$

It is our hope that this assertion can be proven for some wide class of maps, perhaps all maps that have a hyperbolic repelling set. However, we do not know any general proof of statement **1.4**.

Lacking such a theorem, we must examine some more fragmentary evidence. In the next section, we reformulate condition **1.4** and list a variety of simple cases in which this condition is known to be satisfied. We then notice that all strange repellers known to satisfy Eq. **1.4** arise in a situation in which there is a hyperbolic, but fully repulsive, set. For this reason, we examine in *Section 3* a map with both expansion and contraction (7) and show that the basic result **1.4** equally holds in this case.

2. Formulation

The definition of Γ_n in terms of Lebesque measures can be converted into an integral statement, namely

$$\Gamma_n = \int_R d\mathbf{r}' \int_R d\mathbf{r}\,\delta[\mathbf{r}' - f^n(\mathbf{r})] \Big/ \int_R d\mathbf{r}''. \qquad [2.1a]$$

An analogous expression for A_n is obtained from Eq. **1.2**. If all the unstable cycles of f^n lie in R, we have

$$A_n = \int_R d\mathbf{r}\,\delta[\mathbf{r} - f^n(\mathbf{r})]. \qquad [2.1b]$$

To understand the asymptotic relation between these two

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expressions, introduce the Frobenius–Perron operator P, defined as operating to the left on states $\langle\phi|$. If the state $\langle\phi|$ corresponds to a function $\phi(\mathbf{r})$, then the state $\langle\phi|P$ has

$$\langle\phi|P = \langle\psi| \quad [2.2]$$

if

$$\psi(\mathbf{r}) = \phi[f(\mathbf{r})].$$

Now let us assume that the set R has the very special property that if $\mathbf{r}_0$ lies within R but $\mathbf{r}_1 = f(\mathbf{r}_0)$ does not, then $\mathbf{r}_n = f^n(\mathbf{r}_0)$ will not be elements of R for each value $n = 2, 3, \ldots$. If all this is true, we can comfortably define an inner product

$$\langle\phi_1|\phi_2\rangle = \int_R d\mathbf{r}\phi_1(\mathbf{r})\phi_2(\mathbf{r}) \quad [2.3]$$

and then write expressions **2.1** as

$$\Gamma_n = \langle 0|P^n|0\rangle/\langle 0|0\rangle \quad [2.4a]$$

$$A_n = \text{trace } P^n. \quad [2.4b]$$

Here $\langle 0|$ is a state with $\phi_0(\mathbf{r}) = 1$ and P^n is an operator with a matrix realization $\langle\mathbf{r}'|P^n|\mathbf{r}\rangle = \delta[\mathbf{r}' - f^n(\mathbf{r})]$, as is obtained by multiplying P n times.

If P were a function of a hermitian operator, all the rest would be easy. Then P would have a set of eigenvalues $\exp(-\varepsilon_\mu)$, $\mu = 0, 1, 2, \ldots$. The eigenvalue (say the one with $\mu = 0$) having the smallest real part of ε_μ would dominate for large n. In the large n limit, Eqs. **2.4** would reduce to

$$\Gamma_n \rightarrow \gamma\, e^{-n\varepsilon_0}$$

$$A_n \rightarrow I_0\, e^{-n\varepsilon_0}. \quad [2.5]$$

Here I_0 is the multiplicity of the hypothetical lowest lying state. Then Eq. **2.5** would guarantee the correctness of our basic result (**1.4**), with α being given by the lowest eigenvalue ε_0.

But this last paragraph is a pipe dream because P is certainly not known to be a function of a hermitian operator. But, we can get closer to the results (**2.5** and **1.4**) if the mapping f is hyperbolic and has a Markov decomposition. In that case, one can use one-dimensional statistical mechanics to show (8) that A_n does indeed have a kind of property analogous to a spectral decomposition—namely, that one can write

$$A_n = \sum_\mu I_\mu\, e^{-n\varepsilon_\mu}, \quad [2.6]$$

where the I_μs are integers but are not necessarily positive. When the eigenvalues are widely spaced, representation **2.6** is a great help in obtaining accurate estimates of ε_0 based on values of A_n for relatively low n.

One case in which Eq. **1.4** is certainly valid is the one in which the region R is a small neighborhood of a hyperbolic fixed point, $\mathbf{r}_0$. Let $\Lambda = \mathrm{D}f(\mathbf{r}_0)$ be the derivative matrix at the fixed point. Then, according to Eq. **1.2**,

$$A_n = |\det(1 - \Lambda^n)|^{-1}. \quad [2.7]$$

For large n, we need consider only the eigenvalues of the matrix Λ that lie outside the unit circle. If these eigenvalues are of the form $\Lambda' = e^{\delta_l}$, Eqs. **2.7** and **1.3** give

$$\delta = \sum_l \delta_l. \quad [2.8]$$

Here δ_ls are the logarithms of the eigenvalues of the Floquet matrix, Λ, and the sum is constrained by the condition Re $\delta_l > 0$. Thus only the expanding directions enter.

To see that Eq. **2.8** is right for the fixed point, visualize a case with one expanding direction, "x," and one contracting direction, "y," and let R be a rectangle with sides parallel to x and y (Fig. 1). The image of this rectangle is the region R'. One can calculate Γ_1 (which is $e^{-\alpha}$) as the ratio of the area of overlap between R and R' to the total area of R'. This ratio gives the staying probability. But notice that the ratio does depend on the expansion rate but is independent of the contraction rate. This argument then establishes the result $\alpha = \delta$ for this kind of fixed point.

The case in which $\mathbf{r}$ is a number so that f is a mapping on the real line was considered in a previous publication (16). The situation in which $f(x) = x^2 + p$ with x and p real and $p < -2$ was investigated in detail. In this case, the strange repeller is a Cantor set. The quantities Γ_n could be accurately evaluated by using the inverse images of $x = 0$, while A_n could be found from the cycles. The speculation $\alpha = \delta$ was substantiated by this calculation, at least for the example in question.

This same publication also considered the complex version of this mapping problem, $f(z) = z^2 + p$. Now the repeller is a Julia set (9). Numerical and analytical arguments were presented that strongly suggested that $\alpha = \delta$ in this case. The analytic nature of the map leads to a simpler form of Eq. **1.2**—namely,

$$A_n = \sum_{z \in \mathrm{Fix}\, f^n} \frac{1}{\left|1 - \dfrac{df^n}{dz}(z)\right|^2}. \quad [2.9]$$

In these two examples of escape from strange sets, all the cycles have only repulsive directions. There is no attraction. To increase our range of experience with Eq. **1.4**, we consider in the next section a case in which there is both an expanding and a contracting direction.

3. Escape Rates on a Two-Dimensional Manifold

In an earlier paper (7), a localization problem led the consideration of the escape rate for a mapping in a three-dimensional space in which $\mathbf{r} = (x, y, z)$ and

$$f((x, y, z)) = (2xy - z, x, y). \quad [3.1]$$

It turns out that the mapping has a simple "time-reversal" symmetry. If you have an orbit $\mathbf{r}_{j+1} = f(\mathbf{r}_j)$ that has x-values $\ldots x_j, x_{j+1}, x_{j+2} \ldots$, then one has an equally good solution with x-values $\ldots x_{j+2}, x_{j+1}, x_j \ldots$. As a result, for every cycle,

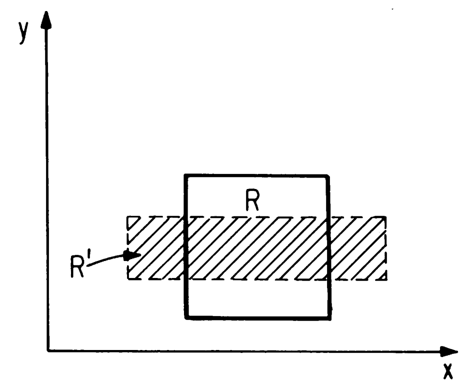

FIG. 1. A mapping with one direction (x) expanding and the other direction (y) contracting.

Table 1. Comparison of α and δ for the mapping **3.1**

n	0	0.5	1	10	50
2	0		0.183919		
4	0		0.286357		
5	0		0.306945		
6	0		0.267650		
7	0		0.301200		
8	0		0.300088		
9	0		0.311232		
10	0	0.13794706	0.300361	1.3986	2.226
11	0	0.13796868	0.300457	1.4140	2.293
12	0	0.14021587	0.298761	1.4026	2.235
13	0	0.13795662	0.300434	1.4107	2.281
14	0	0.13795567	0.300426	1.4048	2.242
15	0	0.13821219	0.301026	1.4092	2.273
16	0	0.13795594	0.30042795	1.4059	2.246
17	0	0.13795606	0.30042868	1.4983	—
∞		0.13795600 ± 0.00000006	0.3004283 ± 0.0000003	1.4071 ± 0.0012	2.26 ± 0.014
α		0.13796 ± 0.00005	0.3003 ± 0.0005	1.42 ± 0.02	2.29 ± 0.1
1/2 ln 2λ				1.50	2.30

Values of $\delta_n = -\ln A_n/n$ as a function of λ and n are presented. The line labeled infinity gives the extrapolated values of δ_∞, with a quoted error that is the half difference between the last two δ_n shown, which may not be conservative because δ_n has larger variations when $n = 0$ (mod 3) than for other values of n. The α-values given are a direct calculation of escape rate, with a statistical quoted error. Note the satisfactory agreement between δ_∞ and α. For completeness, the values of the asymptotic estimate $\delta \sim (1/2) \ln 2\lambda$ ($\lambda \to \infty$), which was derived in Eq. **A.1**, are also given.

if there is a Floquet multiplier outside the unit circle, there is also one inside.

There is a conserved quantity associated with the mapping (**3.1**). Form the combination:

$$\lambda^2(\mathbf{r}) = \lambda^2((x, y, z)) = x^2 + y^2 + z^2 - 1 - 2xyz. \quad \textbf{[3.2]}$$

A brief calculation shows that

$$\lambda^2[\mathbf{f}(\mathbf{r})] = \lambda^2(\mathbf{r})$$

so that combination is unchanged in the course of the mapping.

For this reason, we do not consider an escape problem in the entire three-dimensional euclidean space R^3 but instead focus our attention on manifolds in which the quantity **3.2** is fixed. We focus on the case in which the fixed value of the right-hand side of Eq. **3.2** is a number greater than or equal to zero, so that we can define our manifold by giving a real value of λ between 0 and ∞.

Notice that the manifold in question is certainly not compact. Topologically, it is similar to the surface of a sphere with four arms coming out of the sphere and moving out toward infinity. The manifold contains points with x, y, and z all very large but with the requirement that the product xyz be positive.

In addition to these regions at infinity, the manifold contains a central region where x, y, and z are all of order unity—assuming that λ itself is of order unity. If a point is placed "at random" within the central region, it is very likely that after a few iterations its coordinates will start to grow with greater than exponential rapidity. When this happens we say that a point "has escaped."

To make this definition more precise notice that the recursion relation $\mathbf{r}_{j+1} = f(\mathbf{r}_j)$ may be combined with Eq. **3.1** to give the relation

$$\mathbf{r}_j = (x_j, x_{j-1}, x_{j-2}) \quad \textbf{[3.3]}$$

and the statement

$$x_{j+2} + x_{j-1} = 2x_jx_{j+1} \quad \textbf{[3.4]}$$

From Eq. **3.4**, one can prove (10) that escape to infinity will occur whenever

$$|x_j| > 1$$
$$|x_{j-1}| > 1$$
$$|x_{j-1}|\,|x_j| > |x_{j-2}|. \quad \textbf{[3.5]}$$

We then use conditions **3.5** as our requirement for asserting that $\mathbf{r}_j$ "has escaped."

In ref. 7, we chose a set of initial points with x and y each uniformly distributed between -1 and 1. We then calculated the escape rate α via Eq. **1.1** (Table 1).

The other approach involves finding cycles of the mapping. A very straightforward way for doing this is presented in ref. 10.

To work out A_n in the two-dimensional constant λ manifold, one may first calculate the derivative matrix $\mathrm{D}\mathbf{f}^n(\mathbf{r})|_{\mathbf{r}=\mathbf{f}^n(\mathbf{r})}$ of Eq. **3.1** in three-dimensional euclidean space by a simple matrix multiplication. The eigenvalues of the matrix are of the form 1, η, $(-1)^n/\eta$. The last two of them are also the eigenvalues of the derivative matrix tangent to the λ manifold. Once we know A_n, Eq. **1.3** gives the value of δ. In Table 1 are shown the results for different ns and λs. Notice that for large n, δ and α agree within error.

Appendix

Here we discuss the derivation of the formula

$$\delta = \frac{1}{2} \ln 2\lambda \qquad (\lambda \to \infty). \quad \textbf{[A.1]}$$

Ref. 10 describes a simple way to calculate all the cycles for $\lambda \geq 0$. Any n-length cycle can be expressed symbolically by writing a string containing n symbols from the set L, S, and $\bar{L}$. For large λ, these stand respectively for a value of x of order λ, of order unity, and of order $-\lambda$. If B stands for L or $\bar{L}$, the permitted strings include all possibilities save ones in which two Bs are adjacent, or there are three Ss in a row, or in which the combinations $LSSL$ or $\bar{L}SS\bar{L}$ appear. Each permitted string of length n that does not repeat itself corresponds to one and only one cycle of length n.

We can represent points on the constant-λ manifold by giving the value of x_j, x_{j-1} and an additional quantity $\varepsilon_{j-1/2}$—with values plus or minus one.

In terms of the auxiliary quantity

$$Z_{j-1/2} = \sqrt{\lambda^2 + (1 - x_j^2)(1 - x_{j-1}^2)}\,,$$

the mapping can then be written as

$$(u', v', \varepsilon') = T(u, v, \varepsilon) \qquad \textbf{[A.2]}$$

with

$$\begin{aligned} u' &= uv - \varepsilon z(u, v) \\ v' &= u \\ \varepsilon' &= \mathrm{sign}(v - u''). \end{aligned} \qquad \textbf{[A.3]}$$

The derivative matrix has the simple form

$$M_j = \frac{\partial(u_{j+1}, v_{j+1})}{\partial(u_j, v_j)} = \begin{bmatrix} \dfrac{u_j - u_{j+1}v_j}{\varepsilon_{j-1/2}Z_{j-1/2}} & \dfrac{\varepsilon_{j+1/2}Z_{j+1/2}}{\varepsilon_{j-1/2}Z_{j-1/2}} \\ 1 & 0 \end{bmatrix} \qquad \textbf{[A.4]}$$

and for an n-length cycle

$$\frac{\partial(u_n, v_n)}{\partial(u_0, v_0)} = M_{n-1} \cdot M_{n-2} \cdot \ldots \cdot M_0 \equiv M^n. \qquad \textbf{[A.5]}$$

Let η, $(-1)^n/\eta$ be the eigenvalues of M^n. In the limit $\lambda \to \infty$, Eqs. **A.2**, **A.4**, and **A.5** may be analyzed to give

$$|\eta| \propto \lambda^m, \qquad \textbf{[A.6]}$$

where m is the number of Ss in the cycle. Particularly, for an even-length cycle of the type $BSBS \ldots BS \ldots BS$ (alternate B and S, $m = n/2$), we can show that

$$|\eta| = 2^n \lambda^{\frac{n}{2}} \qquad (\lambda \to \infty). \qquad \textbf{[A.7]}$$

Since $|\eta| \gg 1$,

$$A_n = \sum_{r \in \mathrm{Fix} f^n} \frac{1}{|\eta|}. \qquad \textbf{[A.8]}$$

For large λ, the contributions of the cycles with smallest m (the type described above) dominate. A little counting work shows that the total number of such cycle elements is $2^{\frac{n}{2}+1}$. Therefore,

$$A_n = \frac{2^{\frac{n}{2}+1}}{2^n \lambda^{\frac{n}{2}}} = 2(2\lambda)^{-\frac{n}{2}} \qquad (\lambda \to \infty) \qquad \textbf{[A.9]}$$

and

$$\delta = \frac{1}{2} \ln 2\lambda$$

where Eq. **1.3** has been used.

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