

Dirac operators coupled to vector potentials

(elliptic operators/index theory/characteristic classes/anomalies/gauge fields)

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ABSTRACT Characteristic classes for the index of the Dirac family \not{D}_A are computed in terms of differential forms on the orbit space of vector potentials under gauge transformations. They represent obstructions to the existence of a covariant Dirac propagator. The first obstruction is related to a chiral anomaly.

In this note we study the null spaces (zero frequency modes) of \not{D}_A , the massless Dirac operator coupled to a vector potential, as the potential A varies. We are interested in the null spaces of positive chirality as opposed to those of negative chirality. Their formal difference is a virtual bundle, $\text{Ind } \not{D}$; we apply the index theorem for families of operators and some infinite dimensional geometry to compute the characteristic classes of $\text{Ind } \not{D}$ explicitly in terms of differential forms.

The formulas obtained may be of interest in quantum chromodynamics. The path integral formulation uses gauge invariant functionals of the propagator for \not{D}_A . To define the propagator \not{D}_A^{-1} requires some consistent identification of the null spaces of positive and negative chirality. The nonvanishing of the characteristic classes are obstructions to a consistent covariant identification of these null spaces—i.e., obstructions to the existence of a covariant propagator. The first such obstruction is related to a chiral anomaly, as discussed below. We ask whether the higher obstructions have physical significance as well.

Let M be a compact oriented Riemannian spin manifold of dimension $2n$, and P a principal bundle over M with group G . Let \mathfrak{A} be the set of connections or vector potentials on P , with \mathcal{G} the group of gauge transformations of P . We denote the action of $\phi \in \mathcal{G}$ on $A \in \mathfrak{A}$ by $\phi \cdot A$. Let ρ be a representation of G on \mathbb{C}^N giving the associated vector bundle $E = P \times_G \mathbb{C}^N$. Each $A \in \mathfrak{A}$ gives a Dirac operator $\not{D}_A: C^\infty(S^+ \otimes E) \rightarrow C^\infty(S^- \otimes E)$ where S^\pm are the spin bundles over M of positive and negative chirality, respectively. In local coordinates

$$\not{D}_A = \sum_{\mu=1}^{2n} \gamma_\mu (\partial_\mu + \Gamma_\mu + A_\mu) \left(\frac{1 + \gamma_5}{2} \right)$$

where Γ_μ is the Riemannian connection and acts on spinorial indices, while A_μ acts on the scalar indices $1, \dots, N$. We have the covariance $\not{D}_{\phi \cdot A} = \phi^{-1} \not{D}_A \phi$.

The analytic index of the Dirac family $\{\not{D}_A\}_{A \in \mathfrak{A}}$, which we denote by $\not{D}_{\mathfrak{A}/\mathcal{G}}$ is the formal difference $\{\ker \not{D}_A\}_{A \in \mathfrak{A}} - \{\ker \not{D}_A^*\}_{A \in \mathfrak{A}}$. Each term is not a vector bundle over \mathfrak{A} because the dimensions of $\ker \not{D}_A$ and $\ker \not{D}_A^*$ can jump (the same amount) as A varies over \mathfrak{A} . Nevertheless, the formal difference is well defined as an element of $K(\mathfrak{A})$. Moreover, because of the covariance of \not{D}_A , $\ker \not{D}_{\phi \cdot A} = \phi(\ker \not{D}_A)$, and the formal difference is an element of $K(\mathfrak{A})$ equivariant under \mathcal{G} . In our case it descends to an element of $K(\mathfrak{A}/\mathcal{G})$ which we denote by $\text{Ind } \not{D}$.

The analytic family indexed by \mathfrak{A}/\mathcal{G} can be defined directly in terms of the Hilbert bundles $\mathcal{H}^\pm = \mathfrak{A} \times_{\mathcal{G}} L_2(S^\pm \otimes E)$ over \mathfrak{A}/\mathcal{G} . Covariance means $\{\not{D}_{\phi \cdot A}\}_{\phi \in \mathcal{G}}$ gives an elliptic operator $\not{D}_{\mathfrak{A}/\mathcal{G}}$ mapping the fiber $\mathcal{H}_{\phi \cdot A}^+$ to $\mathcal{H}_{\phi \cdot A}^-$. The analytic index of this family is $\text{Ind } \not{D}$ above.

When $M = S^4$ and \mathcal{G} is the group of gauge transformations leaving the north pole fixed, the index for the Dirac family $\not{D}_{\mathfrak{A}/\mathcal{G}}$ is computed topologically in ref. 1. The index theorem implies that the following two maps are homotopically equivalent. The first is given by the Dirac family

$$\{\phi \cdot A\} \xrightarrow{\not{D}} \not{D}_{\phi \cdot A}$$

mapping \mathfrak{A}/\mathcal{G} into Fredholm operators. For the second, we have the composition of maps

$$\mathfrak{A}/\mathcal{G} \xrightarrow{\alpha_1} \Omega^3(G) \xrightarrow{\alpha_2} \Omega^3(U(N)) \xrightarrow{\alpha_3} \Omega^3(U(\infty)) \xrightarrow{\alpha_4} \mathcal{F}.$$

The map α_1 (which is a homotopy equivalence) is parallel transport by means of A around closed curves parameterized by the equator S^3 . (Follow a fixed geodesic from the north pole to the south pole and follow a variable geodesic back.) The map α_2 is induced by the representation $\rho: G \rightarrow SU(N)$, and α_3 by the injection of $U(N) \rightarrow U(\infty)$. Finally, α_4 is a homotopy equivalence (Bott periodicity, twice).

Thus, the characteristic classes of $\text{Ind } \not{D}$ can be obtained by pulling back the cohomology generators in \mathcal{F} via the second map. For example, if $G = U(N)$ and ρ is the identity, one obtains nonzero characteristic classes, up to degree $2N - 4$.

In general, to compute the characteristic classes of $\text{Ind } \not{D}$ in terms of forms, we introduce a “universal” bundle with connection. \mathcal{G} acts on $P \times \mathfrak{A}$ by $(p, A) \rightarrow (\phi(p), \phi \cdot A)$. This action has no fixed points and gives a principal bundle

$$\left(P \times \mathfrak{A}, \mathcal{G}, \frac{P \times \mathfrak{A}}{\mathcal{G}} = \mathfrak{Q} \right).$$

Since the group action of G on $P \times \mathfrak{A}$ commutes with that of \mathcal{G} , the group G acts on \mathfrak{Q} . If G acts without fixed points, one obtains a principal bundle \mathfrak{Q} with group G and base $\mathfrak{Q}/G = M \times \mathfrak{A}/\mathcal{G}$. That occurs when one either restricts \mathfrak{A} to the space of irreducible connections or restricts \mathcal{G} to be gauge transformations leaving a point of P fixed. We assume the latter. The principal G -bundle \mathfrak{Q} has a natural connection w , obtained as follows. The space $P \times \mathfrak{A}$ has a Riemannian metric invariant under $G \times \mathcal{G}$. At (p, A) , the metric on $T(P, p)$ is given by the metric of G, M and the connection A ; while the metric on $T(\mathfrak{A}, A)$ is the usual metric on $C^\infty(\Lambda^1 \otimes g)$. The metric on $P \times \mathfrak{A}$ descends to a metric on \mathfrak{Q} invariant under G . The orthogonal complement to orbits of G gives the connection w .

(\mathfrak{Q}, w) is universal in the following sense. Suppose Q is a principal G -bundle over $M \times X$, X compact and $Q|_{M \times x} \cong P$ for each $x \in X$. Suppose, moreover, that Q has a fiber connection; that is, a choice of connection on $Q|_{M \times x}$ continuous for $x \in X$. Then there is a map $A: Q \rightarrow \mathfrak{Q}$ inducing the fiber connection from w . Conversely, any map β of $X \rightarrow \mathfrak{A}/\mathcal{G}$ leads to a fiber connection by pulling back (\mathfrak{Q}, w) via $I \times \beta: M \times X \rightarrow M \times \mathfrak{A}/\mathcal{G}$.

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The curvature \mathcal{F} of w is easily computed. It is a horizontal 2-form with values in \mathfrak{g} , the Lie algebra of G , and has components of type (2, 0), (1, 1), and (0, 2) reflecting the product base space $M \times \mathcal{U}/\mathcal{G}$. The formulas for \mathcal{F} at (p, A) are as follows: (1) $\mathcal{F}_{t_1, t_2} = F_{t_1, t_2}(A)$ for $t_1, t_2 \in T(M, \pi(p))$; (2) $\mathcal{F}_{t, \tau} = \tau(t)$ for $t \in T(M, \pi(p))$ and $\tau \in T(\mathcal{U}/\mathcal{G}, \{A\})$ (so that $\tau \in C^\infty(\Lambda^1 \otimes \mathfrak{g})$ and $D_A^* \tau = 0$); and (3) $\mathcal{F}_{\tau, \sigma} = Gb_\tau^*(\sigma)$ where $G = (D_A^* D_A)^{-1}$ and $b_\tau: \Lambda^0 \otimes \mathfrak{g} \rightarrow \Lambda^1 \otimes \mathfrak{g}$ is given by $f \rightarrow [f, \tau]$. In local coordinates, the (2, 0) component of the field is $F_{\mu, \nu}$. The (1, 1) component is δA_μ , while the (0, 2) component is $(D_A^* D_A)^{-1}[\delta A_\mu, \delta B_\nu](x)$ with δA and δB in the background gauge. Here, π is the projection of P onto M .

If we apply the index formula for a family (2) to $\not{D}_{\mathcal{U}/\mathcal{G}}$ one obtains *Theorem 1*.

THEOREM 1. $ch(Ind \not{D}) = \int_M \hat{a}(M) ch(\mathcal{E})$ where $\mathcal{E} = \mathcal{Q} \times_G \mathbb{C}^N$ a vector bundle over $M \times \mathcal{U}/\mathcal{G}$, \hat{a} is the usual characteristic class associated with the spinor index and ch is the Chern character.

The curvature formulas above give explicit formulas for the characteristic classes of \mathcal{E} in terms of differential forms (3). For example, suppose $M = S^{2n}$, $G = SU(N)$, and ρ is the identity representation. Then, $\hat{a}(M) = 1$ and the Chern character of $Ind \not{D}$ is expressed in terms of the Chern classes of \mathcal{E} integrated over M . These Chern classes are the invariant polynomials $k_j(\mathcal{F})$ where

$$\sum k_j(T) t^{N-j} = \det\left(tI_N + \frac{i}{2\pi} T\right).$$

The invariant polynomials $k_j(T)$ are also expressible in terms of $\text{tr}(T^k)$, and there is some simplification for $SU(N)$ since $\text{tr}(T) = 0$. So,

$$\begin{aligned} k_0 &= 1, \\ k_1(T) &= \frac{i}{2\pi} \text{tr}(T) = 0, \\ k_2(T) &= -\frac{1}{8\pi^2} \text{tr}(T^2), \\ k_3(T) &= -\frac{i}{24\pi^3} \text{tr}(T^3), \\ k_4(T) &= -\frac{1}{2^6 \pi^4} \left(\text{tr}(T)^4 - \frac{(\text{tr}(T)^2)^2}{2} \right) \\ &= -\frac{1}{2^6 \pi^4} (\text{tr}(T)^4) + k_2^2(T)/2, \text{ etc.} \end{aligned}$$

COROLLARY 1.1. Let $M = S^{2n}$. The Chern classes of $Ind \not{D}$ are expressible in terms of $d_{2j} = \int_{S^{2n}} k_{j+n}(\mathcal{F})_{2n, 2j}$ forms of degree $2j$ on \mathcal{U}/\mathcal{G} , where $k_{j+n}(\mathcal{F})_{2n, 2j}$ stands for the $(2n, 2j)$ component of $k_{j+n}(\mathcal{F})$.

For example, when $M = S^4$, the 0th Chern class is

$$\int_{S^4} k_2(\mathcal{F})_{4,0} = -\frac{1}{8\pi^2} \int_{S^4} \text{tr}(\mathcal{F}^2)_{4,0} = -\frac{1}{8\pi^2} \int_{S^4} \text{tr}(F^2),$$

the usual Pontrjagin index. While the first Chern class c_1 of $Ind \not{D}$ equals

$$\begin{aligned} \int_{S^4} k_3(\mathcal{F})_{4,2} &= -\frac{i}{24\pi^3} \int_{S^4} \text{tr}(\mathcal{F}^3)_{4,2} \\ &= -\frac{i}{24\pi^3} \int_{S^4} \epsilon_{\alpha\beta\gamma\delta} \text{tr}\{F_{\alpha\beta} F_{\gamma\delta} Gb_\tau^*(\sigma) \\ &\quad + F_{\alpha\beta} Gb_\tau^*(\sigma) F_{\gamma\delta} + F_{\alpha\beta}(\tau_\gamma \sigma_\delta + \sigma_\delta \tau_\gamma)\} \end{aligned}$$

as a 2 form on \mathcal{U}/\mathcal{G} evaluated on the pair of tangent vectors τ, σ .

When $\rho \neq Id$, the above formulas hold with $\rho \cdot \mathcal{F}$ replacing \mathcal{F} and $\text{tr} \rho$ (the trace in the ρ -representation) replacing tr .

Suppose $G = SU(N)$, $M = S^{2n}$, and \mathcal{G} is the group of gauge transformations leaving a point fixed. Then \mathcal{G} is the group of the principal bundle with base \mathcal{U}/\mathcal{G} and total space \mathcal{U} , which is topologically trivial. The Chern classes $d_{2j} = \int_{S^{2n}} k_{j+n}(\mathcal{F})_{2n, 2j}$, which are $2j$ forms on \mathcal{U}/\mathcal{G} , can be lifted to forms on \mathcal{U} that are exact on \mathcal{U} : $d_{2j} = d\beta_{2j-1}$. Moreover, $\beta_{2j-1}|_{\text{Orbit}} = t_{2j-1}$ is a closed $2j-1$ form on \mathcal{G} representing a generator of $H^{2j-1}(\mathcal{G}, \mathbb{R})$ ($N \geq j+n$), modulo products of lower order.

Although β_{2j-1} are determined only up to an exact differential, secondary characteristic classes give explicit formulas for β_{2j-1} and t_{2j-1} in terms of differential forms. Lift $k_{j+n}(\mathcal{F})$ from $M \times \mathcal{U}/\mathcal{G}$ to \mathcal{Q} , where it equals $d_{\mathcal{Q}} \alpha_{2j+2n-1}$, with $\alpha_{2j+2n-1}$ the secondary characteristic class (formula 73 in ref. 3). That is, $\alpha_{2j+2n-1} = \alpha_{2j+2n-1}(w) = (j+n) \cdot \int_0^1 k_{j+n}(w, \mathcal{F}_t, \dots, \mathcal{F}_t) dt$ with $\mathcal{F}_t = t\mathcal{F} + \frac{1}{2}(t-t^2)[w, w]$. Lift $\alpha_{2j+2n-1}$ to $P \times \mathcal{U}$ and denote it by $\tilde{\alpha}_{2j+2n-1}$. For simplicity, assume $P = M \times G$ (the $k=0$ sector) so that $M \times \mathcal{U} \subset P \times \mathcal{U}$. Let $\beta_{2j-1} = \int_M \tilde{\alpha}_{2j+2n-1}$ a $2j-1$ form on \mathcal{U} , and let t_{2j-1} be the restriction of β_{2j-1} to an orbit $\mathcal{G} \cdot A$.

THEOREM 2. $d\beta_{2j-1} = d_{2j}$. When $\rho = Id$, then t_{2j-1} represents a primitive element in $H^{2j-1}(\mathcal{G}, \mathbb{R})$ $j+n \leq N-i.e.$, t_{2j-1} represents a generator modulo products of lower order.

The nonproduct case is slightly more complicated. The G -connection w on \mathcal{Q} comes from a G -connection \tilde{w} on $P \times \mathcal{U}$. Choose a connection B on P and extend it to $P \times \mathcal{U}$. The form $\tilde{\alpha}_{2j+2n-1}$ used above is replaced by α where

$$k_{j+n}(\mathcal{F}_{\tilde{w}}) - k_{j+n}(F_B) = d\alpha$$

and α is given by formula 70 in ref. 3—i.e.,

$$\alpha = (j+n) \int_0^1 k_{j+n}(\tilde{w} - B, \mathcal{F}_t, \dots, \mathcal{F}_t) dt$$

and

$$\mathcal{F}_t = \mathcal{F}_{B+t(\tilde{w}-B)}.$$

It should be remarked that the characteristic classes d_{2j} are not local, for they involve the Green's operator $(D_A^* D_A)^{-1}$ in the curvature \mathcal{F} . However, the closed forms t_{2j-1} on \mathcal{G} are local and *Theorem 2* implies they are directly expressible in terms of the Chern-Simons secondary classes. That is, suppose f_1, \dots, f_{2j-1} are elements in the Lie algebra of \mathcal{G} —i.e., in $C^\infty(\Lambda^0 \otimes \mathfrak{g})$; because \mathcal{G} acts on P , the f s can be viewed as vertical vector fields on $P = M \times G$. They are also left invariant vector fields on \mathcal{G} .

Let $i(f)$ denote interior product by the vector field f and

$$i(f_1, \dots, f_{2j-1}) = i(f_{2j-1}) \dots i(f_1).$$

Then, at $\phi \in \mathcal{G}$,

$$t_{2j-1}(f_1, \dots, f_{2j-1}) = \int_M i(f_1, \dots, f_{2j-1}) \alpha_{2j+2n-1}(\phi A).$$

For example, for $M = S^4$ and $j = 1$, we obtain the 1-form

$$\begin{aligned} t_1(f) &= \int_{S^4} i(f) \alpha_5(\phi A) = -\frac{i}{24\pi^3} \cdot 3 \int_{S^4} i(f) \int_0^1 \text{tr}(\phi A (tF_{\phi A} \\ &\quad + \frac{1}{2}(t-t^2)[\phi A, \phi A]) (tF_{\phi A} + \frac{1}{2}(t-t^2)[\phi A, \phi A]) dt. \end{aligned}$$

This formula for t_1 is the formula for a nonabelian chiral anomaly (4-6). See also refs. 7-9 for a self contained account

of the relationship between anomalies in all dimensions and secondary characteristic classes.

One interpretation for this anomaly involves determinants. Consider the operator $T_\phi = \not\partial_B^* \not\partial_{\phi A}: C^\infty(S^+ \otimes E) \rightarrow C^\infty(S^+ \otimes E)$, when $\not\partial_A$ and $\not\partial_B$ have no zero frequency modes. The operator T_ϕ is a Laplacian plus lower-order term. It has pure point spectrum $\{\lambda_j\}$, and all but a finite number of eigenvalues lie inside a wedge about the positive real axis. Hence, $\Sigma \lambda_j^{-s}$ makes sense except for a finite number of eigenvalues lying on the negative real axis.

When T has positive eigenvalues one can define $\log \det T$ as

$$-\frac{d}{ds} \Big|_{s=0} \text{tr}(T^{-s}).$$

We extend this definition by letting $I-P$ denote projection on a finite dimensional space spanned by the eigenfunctions having eigenvalues $\lambda_1, \dots, \lambda_k$, including those eigenvalues in $[-\infty, 0]$. Let

$$\det T_\phi = e^{\log \det (PT_\phi)} \prod_{j=1}^k \lambda_j,$$

which is well defined. Moreover, $\phi \rightarrow \det T_\phi$ is a smooth nonvanishing complex valued function on \mathcal{G} . Since $\log \det T_\phi$ may not be definable, $\det T_\phi$ can give a nontrivial element in $H^1(\mathcal{G}, \mathbb{R})$. A direct computation using ζ function regularization gives *Theorem 3*.

THEOREM 3. $t_1 = \frac{1}{2\pi i} d(\det T_\phi)/\det T_\phi + df$; that is, t_1 and $\frac{1}{2\pi i} d(\det T_\phi)/\det T_\phi$ represent the same element of $H^1(\mathcal{U}/\mathcal{G}, \mathbb{R})$.

As explained above, the 1-form t_1 on \mathcal{G} comes from the first Chern class d_2 of $\text{Ind } \not\partial_{\mathcal{U}/\mathcal{G}}$, a 2-form on \mathcal{U}/\mathcal{G} equaling $\int_{S^4} k_3(\mathcal{F})_{4,2}$. The first Chern class of $\text{Ind } \not\partial$ is the Chern class of the determinant line bundle of $\text{Ind } \not\partial$, and it has the following physical interpretation. Consider the fermionic path integral

$$\mathcal{F}_r(A) = \int e^{\bar{\psi} \not\partial_A \psi} \bar{\psi}(y_1) \psi(x_1) \bar{\psi}(y_2) \psi(x_2) \dots \bar{\psi}(y_r) \psi(x_r) \mathcal{D}\bar{\psi} \mathcal{D}\psi$$

which equals

$$\sum_{\pi} (-1)^\pi (\det \not\partial_A^* \not\partial_A)^{1/2} \{E_A(y_{\pi(1)}, x_1) E_A(y_{\pi(2)}, x_2) \dots E_A(y_{\pi(r)}, x_r)\},$$

π is a permutation and E_A is the propagator for $\not\partial_A$. Expand E_A in terms of the eigenvectors ψ_j of $\not\partial_A^* \not\partial_A$ and $\bar{\psi}_j = \not\partial_A \psi_j / \lambda_j$ of $\not\partial_A \not\partial_A^*$ obtaining

$$\sum \frac{\bar{\psi}_j(y) \otimes \psi_j(x)}{\lambda_j}.$$

In particular

$$\int e^{\bar{\psi} \not\partial_A \psi} \bar{\psi}(y) \psi(x) = (\prod \lambda_j) \sum \frac{\bar{\psi}_j(y) \psi_j(x)}{\lambda_j},$$

all quantities depending on A . The expression makes sense when there are no zero frequency modes. Suppose $A \rightarrow B$ with $\lambda_1 \rightarrow 0$. The expression $\mathcal{F}_1(A)$ approaches $\det/\lambda_1 \cdot \bar{\psi}_1(y) \otimes \psi_1(x)$ with $\not\partial_B \psi_1 = 0$. However, ψ_1 is determined only up to a phase and a consistent choice must be made.

For $r > 1$, it is easy to see that because of the exclusion principle, $\mathcal{F}_r(B)$ is indeterminate only when there are exactly r zero frequency modes for $\not\partial_B$. Moreover, the indeter-

minacy depends only on a phase, the choice of a generator in $\Lambda^r(\ker \not\partial_B)$, the 1-dimensional space of skewsymmetric r tensors of $\ker \not\partial_B$.

THEOREM 4. A gauge covariant $\mathcal{F}_r(A)$ smooth in A exists if and only if the determinant line bundle of $\text{Ind } \not\partial$ is trivial—i.e., $d_2 = 0$ in $H^2(\mathcal{U}/\mathcal{G}, \mathbb{Z})$ or $t_1 = 0$ in $H^1(\mathcal{G}, \mathbb{Z})$.

The characteristic forms $d_{2j} \in H^{2j}(\mathcal{U}/\mathcal{G}, \mathbb{Z})$ are obstructions to the existence of a covariant propagator for $\not\partial_{\mathcal{U}/\mathcal{G}}$. We ask the question: Do the higher obstructions have physical significance?

Using our earlier discussion of the topological index, one can show, for $M = S^{2n}$ and $G = SU(N)$, *Theorem 5*.

THEOREM 5. If ρ is the identity representation, then $d_{2j} \in H^{2j}(\mathcal{U}/\mathcal{G}, \mathbb{R})$ and $t_{2j-1} \in H^{2j-1}(\mathcal{G}, \mathbb{R})$ do not vanish for $j \leq N - n$.

Gravitational anomalies are the subject of a recent preprint (10), especially for the Dirac operator, the Rarita-Schwinger operator, and the signature operator. These operators are dependent on the metric and are covariant under diffeomorphisms. The formulas obtained in ref. 10 by perturbative calculations at the one-loop level can also be obtained by the methods described in this paper, using the families index and secondary characteristic classes (unpublished result; O. Alvarez and B. Zumino, personal communication).

Specifically, \mathcal{U} is replaced by the space of all metrics \mathcal{M} of the manifold M . \mathcal{G} is replaced by the group of diffeomorphisms of M leaving a basis at one point fixed ($\text{Diff}_0(M)$). Each metric $\rho \in \mathcal{M}$ gives a Dirac operator $\not\partial_\rho$ (and other geometric operators) with the covariance $\not\partial_{\phi\rho} = \phi^{-1} \not\partial_\rho \phi$ for $\phi \in \text{Diff}_0(M)$. Thus, $\mathcal{M}/\text{Diff}_0(M)$ is the parameter space for the family $\{\not\partial_\rho\}$.

The space $P \times \mathcal{U}$ is replaced by a sub-bundle of $B \times \mathcal{M}$ where B is the bundle of bases of M . The sub-bundle is the set of all frames relative to each metric $\rho \in \mathcal{M}$. The group $\text{Diff}_0(M)$ acts on the sub-bundle and gives a quotient Q which is a principal $O(n)$ bundle over a base space, itself a fiber space over $\mathcal{M}/\text{Diff}_0(M)$ with fiber M . The first Chern class of the family can be promoted to a 1-form on $\text{Diff}_0(M)$, which is directly expressible in terms of secondary characteristic classes. Since only Pontrjagin classes are involved, nonzero results are obtained only in dimensions $n = 4k + 2$.

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Note Added in Proof. An exposition of the first obstruction and its relation to the chiral anomaly, intended primarily for physicists, can be found in ref. 11.

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