Proc. Natl. Acad. Sci. USA
Vol. 86, pp. 8610–8613, November 1989
Mathematics

Solutions to Yang–Mills equations that are not self-dual

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ABSTRACT The Yang–Mills functional for connections on principle $SU(2)$ bundles over S^4 is studied. Critical points of the functional satisfy a system of second-order partial differential equations, the Yang–Mills equations. If, in particular, the critical point is a minimum, it satisfies a first-order system, the self-dual or anti-self-dual equations. Here, we exhibit an infinite number of finite-action nonminimal unstable critical points. They are obtained by constructing a topologically nontrivial loop of connections to which min–max theory is applied. The construction exploits the fundamental relationship between certain invariant instantons on S^4 and magnetic monopoles on H^3. This result settles a question in gauge field theory that has been open for many years.

The existence of non-self-dual solutions to the Yang–Mills equations on S^4 has been an open problem for more than a decade. Evidence for nonexistence rests primarily on the analogy with the problem of harmonic maps from S^2 to S^2, where it is not difficult to show that all solutions to the harmonic map equation are conformal, anticonformal, or trivial. The general philosophy has been that the $SU(2)$ Yang–Mills equations on $S^4 = \mathbb{R}^4 \cup \infty$ are similar, with the quaternions replacing the complex numbers. Partial success has been achieved by Bourguignon and Lawson (1), Bourguignon *et al.* (2), and Taubes (3), who have shown that non-self-dual solutions cannot take on local minima.

In this paper, we show that there exist an infinite number of nontrivial non-self-dual solutions to Yang–Mills equations on S^4 by exploiting the properties of solutions with $U(1)$ symmetries. We produce solutions in the trivial bundle which are left invariant by a $U(1)$ action for every integer $m \geq 2$. The $m = 1$ symmetry describes the basic BPST instanton, and one might hope that the most elementary non-self-dual solution would describe an instanton–anti-instanton balanced pairing with a $U(1)$ symmetry corresponding to $m = 1$. This we have not been able to find. At best we have a procedure for generating a solution that should correspond to an m instanton and m anti-instanton pair for $m \geq 2$.

Our ideas are based on the fundamental relationship between m-equivariant gauge fields on S^4 and monopoles on hyperbolic 3-space $\mathbb{H}^3$, as described by Atiyah (4) [see also Braam (5)]. Taubes (6) has shown the existence of non-self-dual monopoles on Euclidean $\mathbb{R}^3$, and our arguments parallel his but on $\mathbb{H}^3$. By translating our instanton arguments into monopole language, our methods produce hyperbolic monopoles for all real $m > 1$. These monopoles correspond to nonsingular Yang–Mills connections on S^4 exactly when m is an integer. The details of this will appear later.

The analytic techniques in this paper are entirely due to Taubes, who has developed an extensive variational theory for both instantons (3) and Euclidean monopoles (6–9). His analytic arguments carry over exactly to our situation, and we depend on his methods and papers rather than reproducing them here.

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Section 1. The Basic Result

We follow the descriptions of Atiyah and Braam in describing $U(1)$ invariant connections on S^4. Since our equations are conformally invariant, we may change the conformal structure on $S^4 = \mathbb{R}^4 \cup \{\infty\}$. If we consider $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, and introduce polar coordinates in the first factor, we have

$$\mathbb{R}^4 = \{(z, \alpha, (x, y)) : z \geq 0, \alpha \in [0, 2\pi), (x, y) \in \mathbb{R}^2\}.$$

The action of $U(1)$ is

$$q(t)(z, \alpha, x, y) = (z, \alpha + t(\text{mod } 2\pi), x, y).$$

From this coordinate description we obtain the conformal equivalence

$$U(1) \times \mathbb{H}^3 = \mathbb{R}^4 - \{z = 0\} = \mathbb{R}^4 - \mathbb{R}^2$$

or

$$U(1) \times \mathbb{H}^3 = S^4 - S^2.$$

The S^2 is precisely the fixed point set of the action of $U(1)$ on S^4.

To define a $U(1)$ invariant connection on S^4 with structure group $SU(2)$, it is sufficient to give a representation $s: U(1) \to G$ of the symmetry group in the group of gauge transformations and to require that

$$q(t)^*D = s(t)^{-1} \circ D \circ s(t).$$

Since we are working on $\mathbb{R}^4$, we can trivialize on $\mathbb{R}^4$ and assume $s(t) = e^{\hat{i}mt}$ for some integer m. Here $(\hat{i}, \hat{j}, \hat{k})$ will designate a standard basis for $\mathfrak{su}(2)$. This means that in some gauge we can write

$$D = d + \hat{B},$$

where

$$\hat{B} = e^{-\hat{i}m\alpha}(\hat{\phi}d\alpha + A)e^{\hat{i}m\alpha}.$$

Here $q(t)^*A = A$ and $\hat{\phi} \circ q(t) = \hat{\phi}$. This Higgs field $\hat{\phi}$ must be asymptotic to zero as $z \to 0$. The integer m describes the representation class of $U(1) \to SO(3)$ of the symmetry group in the fiber of the gauge group over the fixed point set. We call such connections on S^4 m-equivariant connections. Technically, for m odd we are satisfying the conditions for an equivariant $SO(3)$ solution, which lifts to an $SU(2)$ solution.

The usual gauge change by $s = e^{-\hat{i}m\alpha}$ takes the connection $d + \hat{B}$ on $\mathbb{R}^4$ to the connection $d + B$, where $B = \Phi d\alpha + A$, which is singular along the entire plane $z = 0$. Such planar singularities can be removed exactly when a holonomy condition vanishes, which is equivalent to $m \in \mathbb{Z}$ (10, 11). Here the Higgs field $\Phi = \hat{\Phi} - m\hat{i}$ identifies the integer m via its limit $-m\hat{i}$ as $z \to 0$. Hyperbolic 3-space $\mathbb{H}^3$ is left as the conformal equivalent of $\{z > 0, w = (x, y) \in \mathbb{R}^2\}$, which is the set of dynamic variables for the Higgs field Φ and the connection A. The action for the Yang–Mills functional YM on $D = d + B$ on $\mathbb{R}^4$ is equivalent to a monopole action on $\mathbb{H}^3$.

$$YM(B) = \int_{\mathbb{R}^4} |F^B|^2 (dx)^4$$

$$= 2\pi \int_{\mathbb{R}^2} \int_0^\infty (z^{-1}|D_A\Phi|^2 + z|F^A|^2) dz(dx)^2$$

$$= 2\pi \int_{\mathbb{H}^3} (z^2|D_A\Phi|^2 + z^4|F^A|^2) \frac{dz(dx)^2}{z^3}.$$

The conformal factors $(z)^p$ appear in the norm of a p-form on $\mathbb{H}^3$.

The fundamental part of our calculation consists of the topologically nontrivial loop of m equivariant connections in the trivial bundle over S^4. This is contained in *Section 2*. The second derivatives of the YM functional are uniformly bounded in the natural gauge invariant Sobolev norm in the Hilbert space L_1^2.

$$\|d\,YM(B)\| = \frac{\max \frac{d}{d\varepsilon}|_{\varepsilon=0} YM(B + \varepsilon\,\delta B)}{\|\delta B\|_B}.$$

Here, to make the L_1^2 norm gauge invariant,

$$\|\delta B\|_B^2 = \int_{S^4} (|\nabla^B \delta B|^2 + |\delta B|^2)(dx)^4.$$

This allows us to apply a very general principle of minimizing nontrivial cells in Ljusternik–Schnirelmann theory. This lemma is the same as in finite dimensions, where one proves it by moving finite amounts along the derivative directions. We also work in small balls with equivariant gauges, where $dYM(B_j)$ is also equivariant (see chapters 3 and 4 of ref. 8 for Taubes' discussion).

LEMMA 1. *Let* $q: N \to \mathfrak{U}/G = \mathfrak{B}$ *represent any homotopy class* $K \in [N, \mathfrak{B}]$ *of maps from a compact manifold* N *into the space of* m*-equivariant connections on* S^4. *Let*

$$M = \min_{q \in K} \left(\max_{p \in N} YM(q(p)) \right).$$

Then we can find a subsequence realizing this minimax

$$M = \lim_{j \to \infty} YM(B_j)$$

with

$$\lim_{j \to \infty} \|dYM(B_j)\| \to 0.$$

If the Yang–Mills functional were to satisfy the Palais–Smale condition, we would immediately obtain a solution to Yang–Mills representing every minimax construction (such as our loop) (12). However, we do have a restricted compactness theorem, which is sufficient to obtain our existence theory. The compactness theorem is detailed in *Section 3*.

THEOREM 1. *For every integer* $m \geq 2$, *there exists a nontrivial (nonminimal)* m*-equivariant solution to the Yang–Mills equations in the trivial bundle over* S^4.

Proof: According to *Lemma 2* (*Section 2*), we can construct a topologically nontrivial loop $\Gamma \subseteq \mathfrak{U}/G$ of m-equivariant connections. As we perform a minimization procedure for this loop, by the results of *Theorem 2*, we obtain a sequence of m-equivariant connections B_j satisfying

$$0 < M_m = \lim_{j \to \infty} YM(B_j) < 16\pi^2 m$$

$$\|d\,YM(B_j)\| \to 0.$$

The (strict) lower bound on M_m is obtained, as is Taubes' (6), from elliptic estimates. By the equivariant compactness theorem (*Theorem 3*), we can choose a subsequence converging to an m-equivariant connection away from a finite number of points $(x(1), \ldots, x(n)) \in S^2 \subseteq S^4$ in the fixed point set of $U(1)$. Then

$$B_{j'} \rightharpoonup \tilde{B}.$$

There are two cases. In the first case, $n = 0$, and convergence is in all of S^4. Then

$$YM(\tilde{B}) = M_m$$

and $\tilde{B}$ directly represents the minimax for the constructed loop Γ. We expect this is actually what happens, at least for large m.

An alternative is that there are one or more points $(x(1), \ldots, x(n))$ where this convergence fails. According to our equivariant compactness theorem, associated with each point, via a "blowing-up" procedure, is a nontrivial m-equivariant solution $\tilde{B}$ of the Yang–Mills equation. We accept this solution as a candidate for our nontrivial non-self-dual or anti-self-dual solution. The key point is that these solutions (one or more) cannot lie in nontrivial bundles because there is not enough energy available. We have at most $M_m < 16\pi^2 m$ units of Yang–Mills energy available to us. If any of these solutions lies in a bundle with chern class $c_2 = km \neq 0$, it takes up at least $8\pi^2|k|m \geq 8\pi^2 m$ units. Somewhere it must be cancelled by some topology $c_2 = -km$ so the total adds to zero. This also takes up at least $8\pi^2|k|m \geq 8\pi^2 m$ units. Since we have less than this minimum of $2{\cdot}8\pi^2 m = 16\pi^2 m$ available, all of these blown-up solutions must be in trivial bundles and therefore provide us with our nontrivial m-equivariant solutions to Yang–Mills in the trivial bundle.

In actual fact, the loop we write down possesses a larger symmetry group, $U(1) \times U(1) \times \mathbb{Z}_2$. It would be practical to exploit this entire symmetry in looking for the form of solutions to Yang–Mills more explicitly.

Section 2. A Nontrivial Loop of *m*-Equivariant Connections

Braam (5) follows Chakrabarti (13) and gives an explicit formula for the basic (self-dual) monopole of mass m. In the conformal 3-ball coordinates for $\mathbb{H}^3$, this corresponds to the basic m-equivariant instanton on S^4. We need the asymptotic description of this monopole, which we translate into coordinates in the upper half-space model of $\mathbb{H}^3$ with the monopole concentrated at zero. This describes the asymptotics of the basic m-equivariant instanton concentrated at zero. We reflect this basic m-instanton to a basic m-anti-instanton concentrated at infinity and glue the two together. This construction results in a loop of connections in the trivial bundle if we rotate the angle or gauge of the gluing map. The $m = 1$ instanton is the standard one which looks roughly the same in all gauge directions. However, for $m > 1$, the basic m-instanton (or monopole) has a distinguished direction of self-dual curvature that is asymptotically larger. We match this direction in the instanton–anti-instanton gluing procedure to save or lower energy. The energy that we add in cutting off the other directions is less than what we save by matching the large direction, and the loop corresponds to the gauge rotations that fix the large direction of curvature. We now make this explicit. Fix $m > 1$, so as not to carry around an extra index.

The basic m-monopole is spherically symmetric in $B^3 \simeq \mathbb{H}^3$. It can be written in spherical coordinates (r, ψ, θ) as

$$\Phi = \phi(r)\hat{i}$$

$$A = a(r)(\sin\psi d\theta\hat{j} + d\psi\hat{k}) - \cos\psi d\theta\hat{i}.$$

We change to upper half-plane coordinates (z, ρ, θ) to do the relevant asymptotics and construct the loop. Note that

$$1 - r^2 = \frac{4z}{(1+z)^2 + \rho^2} \qquad \tan\psi = \frac{2\rho}{1 - z^2 - \rho^2}.$$

For $z^2 + \rho^2 = R^2$ small, a computation shows that

$$A = (-\cos\psi)d\theta\hat{i} + O(R^m)$$

$$D_A\Phi = (4zdz)\hat{i} + O(R^m)$$

$$F_A = (4\rho d\rho d\theta)\hat{i} + O(R^{m-1}).$$

Define a cutoff function $\beta(R)$ such that $\beta = 0$ for $R \le \varepsilon$ and $\beta = 1$ for $R \ge 2\varepsilon$, set

$$A_\varepsilon = (-\cos\psi)d\theta\hat{i} + \beta(R)a(r)(\sin\psi d\theta\hat{j} + d\psi\hat{k})$$

and let $\sigma_\varepsilon : \mathbb{H}^3 \to \mathbb{H}^3$ denote reflection in the sphere $R = \varepsilon$.

We now define our loop of configurations $c_\varepsilon^\gamma = (\Phi_\varepsilon^\gamma, A_\varepsilon^\gamma)$, $0 \le \gamma \le \pi$. Let

$$\Phi_\varepsilon^\gamma = \begin{cases} \Phi \text{ for } R \ge \varepsilon \\ \Phi \circ \sigma_\varepsilon \text{ for } R \le \varepsilon \end{cases}$$

$$A_\varepsilon^\gamma = \begin{cases} A_\varepsilon \text{ for } R \ge \varepsilon \\ e^{-\hat{i}\gamma}(\sigma_\varepsilon^* A_\varepsilon)e^{\hat{i}\gamma} \text{ for } R \le \varepsilon. \end{cases}$$

We obtain a loop of connections over S^4 by setting

$$B_\varepsilon^\gamma = A_\varepsilon^\gamma + \Phi_\varepsilon^\gamma d\alpha.$$

LEMMA 2. *For each* $\varepsilon > 0$, *the connections* $d + B_\varepsilon^\gamma$, $\gamma \in [0, 2\pi]$ *form a loop of* L_1^∞ *connections in the trivial bundle that is not contractible in* $\mathfrak{A}/G$.

Proof: B_ε^γ is analytic except along $\rho = 0$, $z = 0$, $R = \varepsilon$. The first two singularities are removed by the gauge transformation $e^{\hat{i}(m\alpha+\theta)}$, and B_ε^γ is Lipschitz continuous across the sphere $R = \varepsilon$. The topological nontriviality of the loop corresponds to the nontriviality of the monopole loop constructed by Taubes (6). The work of Donaldson (14) also contains a multitude of references for the nontriviality of the gauge rotations for gluing instantons onto arbitrary connections.

THEOREM 2. $YM(B_\varepsilon^\gamma) < 16\pi^2 m$ *for* ε *sufficiently small.*

Proof: We briefly sketch the energy estimate over $\mathbb{H}^3$. Let YMH denote the Higgs action over $\mathbb{H}^3$ and YMH_ε the Higgs action over $R \ge \varepsilon$ in $\mathbb{H}^3$. Using conformal invariance, we have $YMH(c_\varepsilon^\gamma) = 2YMH_\varepsilon(c_\varepsilon^\gamma)$. Let $c_0 = (A, \Phi)$ be the basic m-monopole given above and recall that $YMH(c_0) = 4\pi m$. Estimates involving the preceding asymptotics for $m > 1$ imply that

$$YMH_\varepsilon(C_0) \le 4\pi m - C\varepsilon^4 + O(\varepsilon^5)$$

and also that

$$YMH_\varepsilon(c_\varepsilon^\gamma) - YMH_\varepsilon(c_0) = O(\varepsilon^5).$$

We conclude that $YMH(c_\varepsilon^\gamma) < 8\pi m$ for ε sufficiently small. Going to S^4, we obtain *Theorem 2.*

Section 3. Equivariant Weak Compactness Theory

Here we state precisely the theorem that we need. There is an extensive discussion of weak compactness theorems for Yang–Mills theory in the appendix to ref. 8, which applies precisely to our situation. Parker has announced a theory for arbitrary group actions.

THEOREM 3. *Let* B_j *be a sequence of* m-*equivariant connections on* S^4 *satisfying*

$$YM(B_j) \le K,$$

$$\|dYM(B_j)\| \to 0.$$

Then there exists a subsequence $\{j'\}$ *and a finite number of points* $(x(1), \ldots, x(n)) \subseteq S^2$ *such that* $B_{j'}$ *is gauge equivalent to* $B_{j'}'$ *and*

$$B_{j'}' \rightharpoonup \tilde{B}_0$$

weakly in $L_{1,loc}^2(S^4 - \cup_{\alpha=1}^n x(\alpha))$. *For every* $x(\alpha)$, $1 \le \alpha \le n$, *there exists a conformal blow-up* $\sigma(j', \alpha)$ *of a neighborhood and a gauge change such that*

$$\sigma(j', \alpha)^*(B_{j'}') \rightharpoonup \tilde{B}_\alpha$$

in $L_{1,loc}^2(\mathbb{R}^4)$. *In addition* $\tilde{B}_\alpha$, $0 \le \alpha \le n$, *extend to* m-*equivariant solutions of Yang–Mills on* S^4. *Finally* $\tilde{B}_\alpha$, $1 \le \alpha \le n$, *are nontrivial, and if* $n = 0$, ($\tilde{B}_\alpha$ *empty for* $\alpha \ne 0$) *we have*

$$\lim_{j'\to\infty} YM(B_{j'}) = YM(\tilde{B}_0).$$

We remark that Taubes' theory refers back to Sedlacek's weak compactness result (15) and a compactness theorem of Uhlenbeck (16). There are some special properties due to the equivariance. It is easily argued that the missing points $x(\alpha) \in S^2$, since for points in $S^4 - S^2$, we can apply the 3-dimensional compactness theorem (16) directly in $\mathbb{H}^3$ to the connections A_i, where $B_i \simeq (\phi_i, A_i)$ due to the equivariance. This shows the limit connection comes from a hyperbolic monopole on $\mathbb{H}^3$. To extend over the singular set $S^2 \subseteq S^4$, we can argue directly in equivariant neighborhoods using the equivariant good gauge construction (10, 11). Alternatively, we can use the removable codimension two singularity theorem of Sibner and Sibner in the same papers.

Finally, if we construct the conformal blow-ups $\sigma(j', \alpha)$ to commute with the action of S^1, which we are able to do when the centers $x(\alpha) \in S^2$, then the same argument applies to $\sigma(j', \alpha)^* B_{j'}$ on S^4 as did to $B_{j'}$ on S^4. It is a little trickier to obtain the continuity arguments that Taubes obtains, but energy estimates and strict local convergence to Yang–Mills will show that the value of the Yang–Mills functional on the punctured $S^4 - \{\cup_{\alpha=1}^n x(\alpha)\}$ is preserved in the limiting process. From here we obtain the nontriviality of at least one of the Yang–Mills fields in the limit.

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