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Physics

Fractal geometry of music

(physics of melody)

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ABSTRACT **Music critics have compared Bach's music to the precision of mathematics. What "mathematics" and what "precision" are the questions for a curious scientist. The purpose of this short note is to suggest that the mathematics is, at least in part, Mandelbrot's fractal geometry and the precision is the deviation from a log–log linear plot.**

Music until the 17th century was one of the four mathematical disciplines of the quadrivium beside arithmetic, geometry, and astronomy. The cause of consonance, in terms of Aristotelian analysis, was stated to be *numerous sonorus*, or harmonic number. That the ratio 2:1 produces the octave, and 3:2 produces the fifth, was known since the time of Pythagoras. Numerologists of the Middle Ages speculated on the mythical significance of numbers in music. Vincenzo Galilei, father of Galileo, was the first to make an attempt to demythify the numerology of music (1). He pointed out that the octave can be obtained through different ratios of 2^n:1. It is 2:1 in terms of string length, 4:1 in terms of weights attached to the strings, which is inversely related to the cross-section of the string, and 8:1 in terms of volume of sound-producing bodies, such as organ pipes.

Scientific experiments have revealed the relation between note-interval and vibrational frequency produced by an instrument. We obtain an octave-higher note by doubling the sound frequency, which can be achieved by halving the length of a string. There are 12 notes in an octave in our diatonic music; i.e., the frequency difference is divided by 12 equal intervals (i) so that

$$f'/f = (2.0)^{1/12} = 1.05946 = (15.9/15).$$

This relation is well known among musicians, that the ratio of acoustic frequencies between successive notes, f' and f, is approximately 16/15. The ratio of acoustic frequencies I between any two successive music notes of an interval i is

$$I_i = 2^{i/12} = (15.9/15)^i, \qquad [1]$$

where i in an integer, ranging from 1 to 12, in the diatonic music. A semitone is represented by $i = 1$, a tone by $i = 2$, a small third by $i = 3$, etc. The numerical value of I_i is approximately a ratio of integers. Some notes have a ratio of small integers. A fourth ($i = 5$), for example, has an I_5 value of 1.3382, or a ratio of about 4/3; a fifth ($i = 7$) has a value of 1.5036, or a ratio of about 3/2. Those used to be considered consonant tones (1). Others are represented by a ratio of larger integers. A diminished fifth ($i = 6$), for example, has a ratio of 1.4185 (= 10/7.05). This is not a ratio of small integers; it is not even an accurate approximation of 10/7, and this note has been traditionally considered dissonant (1).

Music can be defined as an ordered arrangement of single sounds of different frequency in succession (melody), of sounds in combination (harmony), and of sounds spaced in a temporal succession (rhythm). Melody is supposedly "a series of single notes deliberately arranged in a pattern and chosen from a preexisting series that has been handed down by tradition or is accepted as a convention." Theory of harmony has taught us that the successions of sounds are not random, or that the frequency distribution of i is not chaotic. What is the mathematical expression of this order?

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Fractal Geometry

Studying the frequency of natural catastrophes, one of us (K.J.H.) came to realize an inverse log–log linear relation between the frequency (F) and a parameter expressing the intensity of the events (M), be they earthquakes, landslides, floods, or meteorite impacts (2), and the relation can be stated by the simple equation

$$F = c/M^D. \qquad [2]$$

Only later did we realize that this relation has been called *fractal* by Mandelbrot (3), where c is a constant of proportionality and D is the fractal dimension. Fractal relations have commonly a lower and an upper limit. In the case of earthquakes, for example, Eq. 2 holds only for the interval $3 \leq M \leq 9$, because the smallest earthquakes are not represented by significant statistics, nor is the energy release of large earthquakes infinite.

Mandelbrot (3) put together certain geometric shapes whose "monstrous" forms were very irregular and fragmented; he coined the term fractal to denote them. Those "monsters" were considered irrelevant to nature, akin to modern atonal music (4), until Mandelbrot suggested that the fractal relation could be the central conceptual tool to understand the harmony of nature.

We have been searching for a meaning of melody. Is it tradition or convention, or is it an instinctive expression of a natural law? Could we find a mathematical relation to describe a melody? Could the music of Bach be mathematically distinguished from that of Stockhausen? Could we use mathematics to describe the evolution of music from the primitive folk's music to the atonal music of today? If music is an expression of nature's harmony, could music have a fractal geometry? Which, the atonal or the classical?

Fractal Geometry of Frequency (Melody) in Classical Music

The relative abundance, or the incidence frequency F, of notes of different acoustic frequency f in a musical composition is not fractal (5). Striking keys on a piano does not produce music; melody consists of ordered successions of notes. Are the successions fractal? The relation is fractal if the incidence frequency (F) of the interval (i) between successive notes in a musical composition can be defined by the relation

$$F = c/i^D, \qquad [3a]$$

where D is the fractal dimension of this relation, or

$$\log F = c - D \log i. \qquad [3b]$$

We chose for our first study a composition by J. S. Bach, the first movement of Invention no. 1 in C Major, BWV 772. The percentage incidence frequency F of the interval i between successive notes, whether it be a semitone (i = 1), a tone (i = 2), a small third, . . . , or an octave, has been counted. To analyze the possible difference between the score for the right hand, and the score for the left hand, both have been analyzed and the results are shown by Table 1. There is no significant difference in the distribution pattern in the relation between F and i in the two part scores. We have, therefore, combined both sets of data to evaluate the incidence frequency of the various i in this composition. The result, as shown by Fig. 1*A*, indicates that a fractal relation is established for $2 \leq i \leq 10$, with a fractal dimension $D = 2.4184$:

$$F = 2.15/i^{2.4184}.$$

The notable deviations from the plot are the deficiency of $i = 6$ and the excess of $i = 7$. Although the deviations are small, they are significant to a musician. The diminished fifth, as indicated previously, is an interval not represented by a ratio of small integers. The note is thus difficult to be sung or played accurately on a string instrument, and it used to be called a "devil's note." Composers of classical music have thus consciously avoided this note. The excess of $i = 7$ is also not surprising; the fifth with its 3/2 frequency ratio is considered sonorous and pleasing.

We analyzed then the first movement of Bach's Invention no. 13 in A Minor, BWV 784. This was chosen because Bach tried to impart to each of his 15 inventions a different character, and we wanted to explore the expression of the difference. The fractal relation (Fig. 1*B*) is

$$F = 3.0/i^{1.882}.$$

This relation is valid only for the range $i \geq 3$ and is thus different from the patterns of the other compositions evaluated, which have fractal relations valid for $i \geq 1$ or $i \geq 2$. The unusual deficiency of the full-tone interval (i = 2) is not a chance neglect, but a deliberate measure by Bach to achieve a special effect through the establishment of the small third as the most frequent note interval. The excess of the sonorous fifth (i = 7), another significant deviation from the fractal relation, is also no accident.

The third composition chosen was the adagio movement of Bach's Toccata in F-sharp Minor, BWV 910. A log–log linear plot gives the relation (Fig. 1*C*)

$$F = 0.376/i^{1.3403}$$

for $1 \leq i \leq 7$. The fractal relation is less perfect, probably because this is the most "modern" of the compositions of Bach analyzed. The F/i plot has a pattern intermediate between classical music and modern atonal music. Bach tried to search for something new with this composition. The adagio movement is particularly modern, in the sense that it seems to pose the question, to increase tension before the problem is resolved, and this is apparently achieved through the notable deviations from the fractal relation. There is an unusually large excess of i = 0 (note-repetition), and the excessive repetitions represent a music technique to excite, and to persist. There is a deficiency of i = 4, and the intentional omission of the harmonious great third makes the toccata sound harsh and different.

Leaving Bach, we turned to Mozart and analyzed the first movement of a sonata in F Major by W. A. Mozart, KV 533, and the results are shown by Fig. 1*D*. A fractal relation, well established for the interval $2 \leq i \leq 4$, is

$$F = 1/i^{1.7322}.$$

The notable deviation is the excess of the note i = 5, the sonorous fourth. The deficiency of the diminished fifth (i = 6) is common to all the compositions analyzed.

The constant of proportionality is unity in the fractal geometry of this composition, so that

$$Fi^D = 1. \qquad [4]$$

In this case, D is a similarity dimension (4). This dimension,

$$D = \frac{\log F}{\log(1/i)} = 1.7322,$$

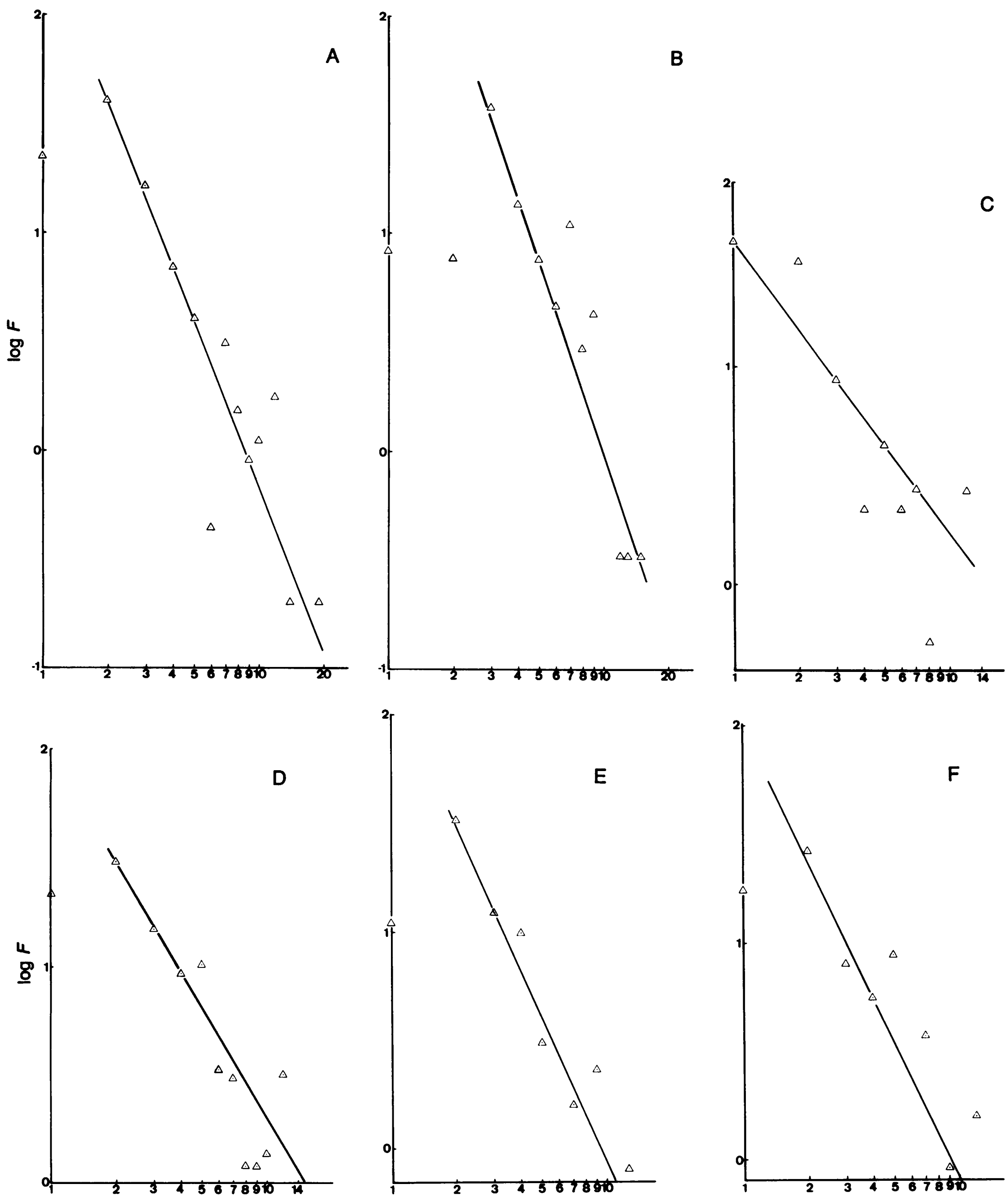
is not an integer and corresponds to nothing in standard geometry. Mandelbrot suggested that a geometrical figure could be generated by this similarity dimension (3). If so, Mozart's music could be considered pictorial, whereas Bach's is precision in mathematics.

To explore the structuring of simple music, we analyzed six Swiss children's songs and grouped the results together in order to obtain significant statistics. The results are shown by Table 2 and Fig. 1*E*. A most notable feature is the excess of note repetition (i = 0). Is this a manifestation that children's music tends to be monotonous?

The theme of folk songs often forms the basis of classical

Table 1. Frequency of incidence of note-intervals in Bach's Invention in C Major

Note-interval	Right hand		Left hand		Total score	
	Incidence	% incidence	Incidence	% incidence	Incidence	% incidence
0	4	1.70	0	0.00	4	0.90
1	50	21.20	51	23.40	101	22.20
2	100	42.40	84	38.50	184	40.50
3	39	16.50	35	16.05	74	16.30
4	20	8.50	11	5.05	31	6.90
5	7	3.00	11	5.05	18	4.00
6	2	0.85	0	0.00	2	0.44
7	5	2.10	9	4.10	14	3.10
8	4	1.70	3	1.40	7	1.54
9	2	0.85	2	0.90	4	0.88
10	2	0.85	3	1.40	5	1.10
11	0	0.00	0	0.00	0	0.00
12	1	0.40	7	3.20	8	1.76
14	0	0.00	1	0.46	1	0.22
19	0	0.00	1	0.46	1	0.22
Total	236	100.00	218	100.00	454	100.00



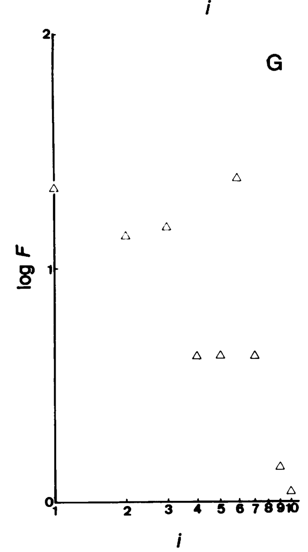


FIG. 1. Fractal geometry of sound frequency. F is the percentage frequency of incidence of note interval i. The approximately linear log–log plots suggest a fractal geometry for the classical music. No such relation is apparent in the music by Stockhausen. (*A*) Bach BWV 772. (*B*) Bach BWV 784. (*C*) Bach BWV 910. (*D*) Mozart KV 533. (*E*) Six Swiss children's songs. (*F*) Mozart KV 331. (*G*) Stockhausen *Capricorn*.

Table 2. Frequency of incidence of note-intervals of music

Note-interval	Bach's Toccata		Swiss children's songs		Mozart's Sonata	
	Incidence	% incidence	Incidence	% incidence	Incidence	% incidence
0	15	8.2	63	24.9	116	27.0
1	69	37.7	28	11.1	76	17.7
2	56	30.6	82	32.4	114	26.6
3	16	8.7	31	12.3	34	7.9
4	4	2.2	25	10.0	24	5.6
5	8	4.4	8	3.1	38	8.9
6	4	2.2	0	0	0	0
7	5	2.7	4	1.6	16	3.7
8	1	0.55	0	0	0	0
9	0	0	6	2.3	4	0.9
10	0	0	4	1.6	0	0
11	0	0	0	0	0	0
12	5	2.7	2	0.8	7	1.6
Total	183	100.0	253	100.1	429	99.9

Table 3. Frequency of notes being played or struck simultaneously (Bach's Toccata in F-sharp Minor)

n*	Played simultaneously		Struck simultaneously	
	Incidence	% incidence	Incidence	% incidence
1	1	1.1	72	45.9
2	2	2.3	38	24.2
3	36	40.9	31	19.7
4	47	53.4	16	10.2
5	2	2.3	0	0
Total	88	100.0	157	100.0

*Number of notes played or struck simultaneously.

music, such as the first movement of Mozart's Sonata in A Major, KV 331. We analyzed the first part of this movement. This composition is characterized by the same excess of note repetition ($i = 0$) as the children's songs (Table 2), and the fractal relations of the two are amazingly similar (*cf.* Fig. 1 *E* and *F*). Not surprising is the total absence of the "devil's note," the diminished fifth.

The fractal relation of Mozart's sonata, imperfect as it is, could also be defined by a similarity dimension: $Fi^D = 1$. Is that a coincidence or is being pictorial a characteristic of Mozart?

To illustrate the obvious difference between classical and modern music, we present in Fig. 1*G* our analysis of Stockhausen's *Capricorn*. There is no resemblance to fractal geometry in this work. The notable deficiency of the great third ($i = 4$) and the extreme excess of the diminished fifth ($i = 6$) are what make modern music atonal.

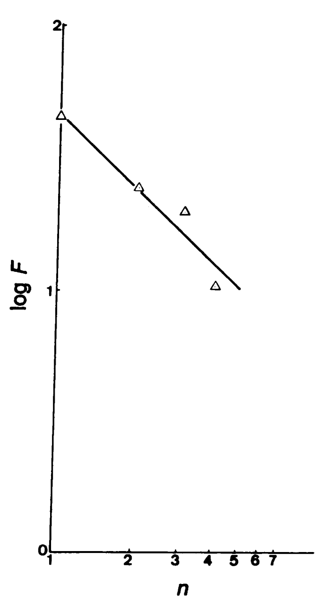


FIG. 2. Fractal geometry of sound amplitude. *F* is the percentage of incidence of *n* notes being struck simultaneously on keyboard. Music is Bach's Toccata, BWV 910.

Fractal Geometry of Amplitude in Bach's Music

Einstein once commented that music consists of acoustic waves, which are definable by their frequency and amplitude. The intervals between successive acoustic frequencies in classical music have a fractal distribution. Is there a similar fractal geometry of the sound as represented by its amplitude?

Voss and Clarke (5) analyzed the loudness of the music and found an approximate fractal distribution of loudness in Bach's first Brandenburg Concerto. They have, however, worked only with an interpretation of Bach by a performer. Was that intended by Bach?

One possible way to evaluate the amplitude is to analyze the number of notes that are played simultaneously, because more notes sounding together should make the sound louder. We chose again the adagio movement of Bach's Toccata, because four melodies are played simultaneously in that fugue. We found that three or four notes from the melodies were sounding simultaneously on 94.3% of the occasions (Table 3). This is not a fractal distribution, nor is this an effective way of evaluating volume.

Recognizing that the intensity of the sound is greatest when a note is first struck on a keyboard, we analyzed the number of notes that are struck simultaneously, and the results of the toccata study are shown by Fig. 2 and Table 3. Although the analysis is based upon only 157 data points, the fractal distribution of amplitude is apparent. The performers of Bach analyzed by Voss and Clarke probably did not distort Bach's original intention.

Summary

This preliminary analysis of a few compositions by Bach and others indicates the potential of making numerical analysis of music. This paper only suggests a methodology, and we have found that the musical effects of a composition can be expressed as deviations from fractal geometry. Nevertheless, many additional random observations would have to be made before one could attempt a profound generalization. One of us (A.J.H.) is now programming software to analyze not only the melody and rhythm but also the harmony of music.

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