Chaos and stability of the solar system

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Over the last two decades, there has come about a recognition that chaotic dynamics is pervasive in the solar system. We now understand that the orbits of small members of the solar system—asteroids, comets, and interplanetary dust—are chaotic and undergo large changes on geological time scales. Are the major planets' orbits also chaotic? The answer is not straightforward, and the subtleties have prompted new questions.

In the early 1600s, Johannes Kepler laid the groundwork for modern celestial mechanics by discovering and formulating the laws of planetary motion from study of the complex observed motions of the planets. Isaac Newton subsequently achieved further simplicity in his mathematical description of the basic laws of motion and the universal law of gravitation. Thus, we came to a simple set of equations that determine the motions of the planets. The force on each planet is simply the sum of the gravitational forces from the sun and all of the other planets in the solar system. In vector notation, this is expressed as:

$$m_i \frac{d^2\mathbf{r_i}}{dt^2} = Gm_i \sum_{j\neq \mathrm{i}} m_j \frac{\mathbf{r_j} - \mathbf{r_i}}{|\mathbf{r_j} - \mathbf{r_i}|^3},$$

where G is the universal constant of gravitation, m values are the masses, and $\mathbf{r}$ values are their positions in space.

For a two-body system (sun and one companion), there is a simple, elegant solution: a conic section (circle or ellipse, parabola or hyperbola). However, the presence of a third body (or in the case of the solar system, the sun, the nine major planets, and myriad minor bodies) allows no simple solution to these simple equations!

Newton himself wrestled unsuccessfully with the problem of the Moon's motion. The main mathematical tool used in this business in the past has been perturbation theory, which starts with an "unperturbed" orbit (an ellipse about the sun) and strives step by step to calculate the effects of perturbations in an orderly fashion. Most uncomfortably, the proof that such a scheme "works" for arbitrarily long times has remained elusive. Do the perturbations cause only small changes in the present nearly circular, nearly coplanar orbits of the planets, or might they add up so much over the great age of the solar system that the orbits change greatly (in the past or future)?

There is the obvious intellectual curiosity. What was the historical planetary configuration? What is it going to be in the future?

There is the mathematical motivation about the quality of solutions of apparently simple ordinary differential equations.

And there are practical motivations—questions related to the habitability of our planet. The climate history of the earth is affected by changes in its orbit, by impacts of asteroids and comets, and possibly also by the accumulation of interplanetary dust particles that are generated by asteroids and comets. Thus the long term orbital dynamics of the planets and small bodies in the solar system has great temporal relevance as well.

We also ask, how typical is our planetary system in the galaxy? What are the characteristics of a *stable* planetary system or of one that harbors a habitable planet?

In the last two decades remarkable advances in digital computer speed, the development of new numerical techniques, and the application of modern nonlinear dynamics techniques and chaos theory to classical problems of celestial mechanics have led to the discovery and exploration of a number of examples of dynamical chaos in our solar system. In its scientific usage, *chaos* is not a synonym for *disorder*, rather it describes the irregular behavior that can occur in deterministic dynamical systems, i.e., systems described by ordinary differential equations free of external random influences. Chaotic systems have two defining characteristics: they show order interspersed with randomness, and their evolution is extremely sensitive to initial conditions. Extreme sensitivity to initial conditions is quantified by the *exponential* divergence of nearby orbits. The rate of such divergence is characterized by the *e*-folding time scale called *Lyapunov time*. A second characteristic time scale is the *escape time*, which is the time for a major change in the orbit.

Chaos in the solar system is associated with gravitational resonances. The simplest case of a gravitational resonance occurs when the orbital periods of two planets are in the ratio of two small integers, e.g., 1:2, 3:5, etc. There are other more subtle gravitational resonances when one considers the precessional periods of planetary orbits in addition to their orbital periods. Strong and weak resonances thread the entire phase space of the solar system in a complex web. *Overlapping resonances*, i.e., multiple gravitational resonances in close proximity, provide the route to chaos in the solar system. Gravitational resonances may effect very large orbital changes or only modest orbital changes (in some cases, even provide protection from large perturbations), depending sensitively on initial conditions. The long term dynamics of the planetary system is the dynamics of gravitational resonances.

We have understood, within the last two decades, that the orbits of many of the small members of the solar system (asteroids, comets, dust particles), subjected to the combined gravitational perturbations of the major planets, are chaotic and unstable on million-year time scales. A dynamical transport mechanism thus has been identified for transporting small bodies across the vast distances in the solar system. This mechanism has led to a large revision in our understanding of the origin of comets and of meteorites.

As an example, Matt Holman described the overlap of high order orbital resonances with Jupiter as well as "three-body resonances" (involving an asteroid's interaction with both Jupiter and Saturn) as the cause of chaos in the outer asteroid belt. Murray and Holman (1) have developed an analytical theory for estimating the Lyapunov time and escape time in this problem.

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The orbital evolution of planetary orbits on giga-year time scales has been investigated recently via several numerical simulations. These have led to a most interesting conclusion that the orbits of the planets themselves evolve chaotically. The characteristic Lyapunov time is 5–10 million years. A new analytic theory (2) shows that chaos among the Jovian planets results from a delicate interaction—also a three-body resonance—among Jupiter, Saturn, and Uranus. The theory also confirms the numerical estimate of the Lyapunov time associated with this chaos and shows that the escape time of Uranus is long (10^{18} years), substantially longer than the lifetime of our sun.

Although the numerical simulations all indicate chaos in planetary orbits, in a qualitative sense the planetary orbits are stable—because the planets remain near their present orbits—over the lifetime of the sun. However, the presence of chaos implies that there is a finite limit to how accurately the positions of the planets can be predicted over long times. Of all of the planets, Mercury's orbit appears to exhibit changes of the largest magnitude in orbital eccentricity and inclination. Fortunately, this is not fatal to the global stability of the whole planetary system owing to the small mass of Mercury. Changes in the orbit of the Earth, which can have potentially large effects on its surface climate system through solar insolation variation, are found also to be small.

Takashi Ito discussed several properties that may be responsible for the long term stability of our solar system. Among these, the difference in *dynamical separation*¶ between terrestrial and Jovian planetary subsystems seems to be quite interesting and important. The terrestrial planets have smaller masses, shorter orbital periods, and wider dynamical separation. They are strongly perturbed by the Jovian planets, which have larger masses, longer orbital periods, and narrower dynamical separation. As a subsystem, the Jovian planets are not perturbed by any other massive bodies.

Ito and Tanikawa (3) have performed a set of numerical experiments to understand how these differences between terrestrial and Jovian planets affect their long term stability. They have considered various kinds of terrestrial planetary subsystems with equal dynamical separations and determined their typical instability time scales under the disturbance from the massive Jovian planets. They find that the terrestrial planetary subsystems with smaller dynamical distances ($<18R_{\mathrm{H}}$) are likely to become unstable in a short time scale ($<10^7$ years). This rapid instability is caused by the strong gravitational perturbation from massive Jovian planets. Thus it seems that the present wide dynamical separation among terrestrial planets ($>26R_{\mathrm{H}}$) is possibly one of the significant conditions to maintain the stability of the planetary orbits in giga-year time spans.

These recent advances are the beginning of a quest to tease out the critical properties of our solar system (and its subsystems) that give it the curious character of being only *marginally chaotic* or *marginally stable* on time spans comparable with its current age. It is but a part of the quest to understand what processes of formation (and perhaps initial conditions) led to this remarkable system in nature and how common such systems are in our galaxy and the universe.

¶For consideration of dynamical stability of two planets of mass m_1 and m_2, orbital radius a_1 and a_2, a natural unit of separation is their mutual *Hill radius* (after the 19th century mathematician, G. W. Hill), defined by

$$R_H = \left(\frac{m_1 + m_2}{3m_{\odot}}\right)^{1/3} \frac{a_1 + a_2}{2},$$

where $m_{\odot}$ is the mass of the sun. With this measure, the separations among terrestrial planets exceed $26R_{\mathrm{H}}$, whereas those among Jovian planets are less than $14R_{\mathrm{H}}$.

1. Murray, N., Holman, M. & Potter, M. (1998) *Astron. J.* **116,** 2583–2589.
2. Murray, N. & Holman, M. (1999) *Science* **283,** 1877–1879.
3. Ito, T. & Tanikawa, K. (1999) *Icarus* **139,** 336–349.

Suggested Readings

1. Malhotra, R. (1999) *Sci. Am.* **281,** 46–53.
2. Peterson, I. (1993) *Newton's Clock: Chaos in the Solar System* (Freeman, New York).

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