The asymptotic distribution of canonical correlations and vectors in higher-order cointegrated models

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The study of the large-sample distribution of the canonical correlations and variates in cointegrated models is extended from the first-order autoregression model to autoregression of any (finite) order. The cointegrated process considered here is nonstationary in some dimensions and stationary in some other directions, but the first difference (the "error-correction form") is stationary. The asymptotic distribution of the canonical correlations between the first differences and the predictor variables as well as the corresponding canonical variables is obtained under the assumption that the process is Gaussian. The method of analysis is similar to that used for the first-order process.

Cointegrated stochastic processes are used in econometrics for modeling macroeconomic time series that have both stationary and nonstationary properties. The term "cointegrated" means that in a multivariate process that appears nonstationary some linear functions are stationary. Many economic time series may show inflationary tendencies or increasing volatility, but certain relationships are not affected by these tendencies. Statistical inference is involved in identifying these relationships and estimating their importance.

The family of stochastic processes studied in this paper consists of vector autoregressive processes of finite order. A vector of contemporary measures is considered to depend linearly on earlier values of these measures plus random disturbances or errors. The dependence may be evaluated by the canonical correlations between the contemporary values and the earlier values.

The nonstationarity of a process may be eliminated by treating differences or higher-order differences (over time) of the vectors. This paper treats processes in which first-order differencing accomplishes stationarity. The first-order difference is represented as a linear combination of the first lagged variable and lags of the difference variable. The stationary linear combinations are the canonical variables corresponding to the nonzero process canonical correlations between the difference variable and the first lagged variable not accounted for by the lagged differences. The number of these is defined as the degree of cointegration.

Statistical inference of the model is based on a sample of observations; that is, a vector time series over some period of time. The estimator of the parameters of the original autoregressive model is a transformation of the estimator of the (stationary) error-correction form. In the latter, one coefficient matrix is of lower rank (the degree of cointegration). It is estimated efficiently by the reduced rank regression estimator introduced by me (1). It depends on the larger canonical correlations and corresponding canonical vectors. The smaller correlations are used to determine the rank of this matrix. Inference is based on the large-sample distribution of these correlations and variables.

The asymptotic distribution of the canonical correlations and coefficients of the variates for the first-order autoregressive process was derived by me (2). The distribution for the higher-order process (that is, several lags) is obtained in this paper, using similar algebra. Hansen and Johansen (3) have independently obtained the asymptotic distribution of the canonical correlations, but by a different method and expressed in a different form.

The likelihood ratio test for the degree of cointegration that I found (1) is given in *Asymptotic Distribution of the Smaller Roots*; its asymptotic distribution under the null hypothesis was found by Johansen (4). To evaluate the power of such a test, one needs to know the distribution or asymptotic distribution of the sample canonical correlations corresponding to process canonical correlations different from 0. See ref. 5, for example.

For further background, the reader is referred to Johansen (6) and Reinsel and Velu (7).

The Model

The general cointegrated model is an autoregressive process $\{\mathbf{Y}_t\}$ of order m defined by

$$\mathbf{Y}_t = \mathbf{B}_1\mathbf{Y}_{t-1} + \cdots + \mathbf{B}_m\mathbf{Y}_{t-m} + \mathbf{Z}_t, \qquad [1]$$

where $\mathbf{Z}_t$ is unobserved with $\mathcal{E}\mathbf{Z}_t = 0$, $\mathcal{E}\mathbf{Z}_t\mathbf{Z}_t' = \mathbf{\Sigma}_{ZZ}$, and $\mathcal{E}\mathbf{Y}_{t-i}\mathbf{Z}_t' = 0$, $i = 1, \ldots$. Let $\mathbf{B}(\lambda) = \lambda^m\mathbf{I} - \lambda^{m-1}\mathbf{B}_1 - \cdots - \mathbf{B}_m$. If the roots $\lambda_1, \ldots, \lambda_{pn}$ of $|\mathbf{B}(\lambda)| = 0$ satisfy $|\lambda_i| < 1$, a stationary process $\{\mathbf{Y}_t\}$ can be defined by **1**. If some of the roots are 1, the process will be nonstationary. In this paper, we assume that n $(0 < n < p)$ roots of $|\mathbf{B}(\lambda)| = 0$ are $1 (\lambda_1 = \cdots = \lambda_p = 1)$, and the other $pm - n$ roots satisfy $|\lambda_i| < 1$, $i = n + 1, \ldots, pm$. The first difference of the process, the "error-correction" form, is

$$\begin{aligned}\mathbf{Y}_t - \mathbf{Y}_{t-1} &= \Delta\mathbf{Y}_t \\ &= \mathbf{\Pi}\mathbf{Y}_{t-1} + \mathbf{\Pi}_1\Delta\mathbf{Y}_{t-1} + \cdots + \mathbf{\Pi}_{m-1}\Delta\mathbf{Y}_{t-m+1} + \mathbf{Z}_t \\ &= \mathbf{\Pi}\mathbf{Y}_{t-1} + \bar{\mathbf{\Pi}}\bar{\Delta}\mathbf{Y}_{t-1} + \mathbf{Z}_t. \qquad [2]\end{aligned}$$

Here $\mathbf{\Pi} = \mathbf{B}_1 + \cdots + \mathbf{B}_m - \mathbf{I} = -\mathbf{B}(1)$, $\mathbf{\Pi}_j = -(\mathbf{B}_{j+1} + \cdots + \mathbf{B}_m)$, $j = 1, \ldots, m-1$, $\bar{\mathbf{\Pi}} = (\mathbf{\Pi}_1, \ldots, \mathbf{\Pi}_{m-1})$, and $\bar{\Delta}\mathrm{Y}_{t-1} = (\Delta Y_{t-1}', \ldots, \Delta\mathbf{Y}_{t-m+1}')'$.

A sample consists of T observations: $\mathbf{Y}_1, \ldots, \mathbf{Y}_T$. Because the rank of $\mathbf{\Pi}$ is k, it is to be estimated by the reduced rank regression estimator introduced by me (1) as the maximum likelihood estimator when $\mathbf{Z}_1, \ldots, \mathbf{Z}_T$ are normally distributed and $\mathbf{Y}_0, \mathbf{Y}_{-1}, \ldots, \mathbf{Y}_{-m+1}$ are nonstochastic and known. The matrices $\mathbf{\Pi}_1, \ldots, \mathbf{\Pi}_{m-1}$ are unrestricted except for the condition $|\lambda_i| < 1$, $i = n + 1, \ldots, pm$. The estimator depends on the canonical correlations and vectors of $\Delta\mathbf{Y}_t$ and $\mathbf{Y}_{t-1}$ conditioned on $\Delta\mathbf{Y}_{t-1}, \ldots, \Delta\mathbf{Y}_{t-m+1}$.

Define

$$\Delta\hat{\mathbf{Y}}_t^+ = \Delta\mathbf{Y}_t - \mathbf{S}_{\Delta Y,\bar{\Delta}Y}\mathbf{S}_{\bar{\Delta}Y,\bar{\Delta}Y}^{-1}\bar{\Delta}\mathbf{Y}_{t-1},$$

$$\hat{\mathbf{Y}}_{t-1}^+ = \mathbf{Y}_{t-1} - \mathbf{S}_{\bar{Y},\bar{\Delta}Y}\mathbf{S}_{\bar{\Delta}Y,\bar{\Delta}Y}^{-1}\bar{\Delta}\mathbf{Y}_{t-1},$$

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where $\mathbf{S}_{\bar{\Delta}Y,\bar{\Delta}Y} = T^{-1}\,\Sigma_{t=1}^{T}\,\bar{\Delta}\mathbf{Y}_{t-1}\bar{\Delta}\mathbf{Y}'_{t-1}$, $\mathbf{S}_{\Delta Y,\bar{\Delta}Y} = T^{-1}\,\Sigma_{t=1}^{T}\,\Delta\mathbf{Y}_t\bar{\Delta}\mathbf{Y}'_{t-1}$, and $\mathbf{S}_{\bar{Y},\bar{\Delta}Y} = T^{-1}\,\Sigma_{t=1}^{T}\,\mathbf{Y}_{t-1}\bar{\Delta}\mathbf{Y}'_{t-1}$. The vectors $\Delta\hat{\mathbf{Y}}_t^{+}$ and $\hat{\mathbf{Y}}_{t-1}^{+}$ are the sample residuals of $\Delta\mathbf{Y}_{t-1}$ and $\mathbf{Y}_{t-1}$ regressed on $\bar{\Delta}\mathbf{Y}_{t-1}$. Define $\hat{\mathbf{S}}^{+}_{\Delta Y,\Delta Y} = T^{-1}\,\Sigma_{t=1}^{T}\,\Delta\hat{\mathbf{Y}}_t^{+}\Delta\hat{\mathbf{Y}}_t^{+\prime} = \mathbf{S}_{\Delta Y,\Delta Y} - \mathbf{S}_{\Delta Y,\bar{\Delta}Y}\mathbf{S}^{-1}_{\bar{\Delta}Y,\bar{\Delta}Y}\mathbf{S}_{\bar{\Delta}Y,\Delta Y}$, $\hat{\mathbf{S}}^{+}_{\Delta Y,\bar{Y}} = T^{-1}\,\Sigma_{t=1}^{T}\,\Delta\hat{\mathbf{Y}}_t^{+}\hat{\mathbf{Y}}_{t-1}^{+\prime} = \mathbf{S}_{\Delta Y,\bar{Y}} - \mathbf{S}_{\Delta Y,\bar{\Delta}Y}\mathbf{S}^{-1}_{\bar{\Delta}Y,\bar{\Delta}Y}\mathbf{S}_{\bar{\Delta}Y}$, and $\hat{\mathbf{S}}^{+}_{\bar{Y}\bar{Y}} = T^{-1}\,\Sigma_{t=1}^{T}\,\hat{\mathbf{Y}}_{t-1}^{+}\hat{\mathbf{Y}}_{t-1}^{+\prime} = \mathbf{S}_{\bar{Y}\bar{Y}} - \mathbf{S}_{\bar{Y},\bar{\Delta}Y}\mathbf{S}^{-1}_{\bar{\Delta}Y,\bar{\Delta}Y}\mathbf{S}_{\bar{\Delta}Y,\bar{Y}}$, where $\mathbf{S}_{\Delta Y,\Delta Y} = T^{-1}\,\Sigma_{t=1}^{T}\,\Delta\mathbf{Y}_t\Delta\mathbf{Y}'_t$, $\mathbf{S}_{\Delta Y,\bar{Y}} = T^{-1}\,\Sigma_{t=1}^{T}\,\Delta\mathbf{Y}_t\mathbf{Y}'_{t-1}$, and $\mathbf{S}_{\bar{Y}\bar{Y}} = T^{-1}\,\Sigma_{t=1}^{T}\,\mathbf{Y}_{t-1}\mathbf{Y}'_{t-1}$. The sample canonical correlations between $\Delta\hat{\mathbf{Y}}_t^{+}$ and $\hat{\mathbf{Y}}_{t-1}^{+}$ and variates are defined by

$$|\hat{\mathbf{S}}^{+}_{\bar{Y},\Delta Y}\hat{\mathbf{S}}^{+\,-1}_{\Delta Y,\Delta Y}\hat{\mathbf{S}}^{+}_{\Delta Y,\bar{Y}} - r^2\hat{\mathbf{S}}^{+}_{\bar{Y}\bar{Y}}| = 0, \qquad \textbf{[3]}$$

$$\hat{\mathbf{S}}^{+}_{\bar{Y},\Delta Y}\hat{\mathbf{S}}^{+\,-1}_{\Delta Y,\Delta Y}\hat{\mathbf{S}}^{+}_{\Delta Y,\bar{Y}}\hat{\gamma} = r^2\hat{\mathbf{S}}^{+}_{\bar{Y}\bar{Y}}\hat{\gamma}, \quad \hat{\gamma}'\hat{\mathbf{S}}^{+}_{\bar{Y}\bar{Y}}\hat{\gamma} = 1. \qquad \textbf{[4]}$$

More information on canonical analysis is covered in chapter 12 of ref. 8. One form of the reduced rank regression estimator is $\hat{\mathbf{\Pi}}_{(k)} = \hat{\mathbf{S}}^{+}_{\Delta Y,\bar{Y}}\hat{\mathbf{\Gamma}}_2\hat{\mathbf{\Gamma}}'_2$, where $\hat{\mathbf{\Gamma}}_2 = (\hat{\gamma}_{n+1}, \ldots, \hat{\gamma}_p)$ and $r_1^2 < \cdots < r_p^2$.

We shall assume that there are exactly n linearly independent solutions to $\boldsymbol{\omega}'\mathbf{B}(1) = 0$; that is, $\omega'\mathbf{\Pi} = 0$. Then the rank of $\mathbf{\Pi}$ is $p - n = k$ and there exists a $p \times n$ matrix $\mathbf{\Omega}_1$ of rank n such that $\mathbf{\Omega}'_1\mathbf{\Pi} = 0$. See Anderson (9). There is also a $p \times k$ matrix $\mathbf{\Omega}_2$ of rank k such that $\mathbf{\Omega}'_2\mathbf{\Pi} = \mathbf{\Upsilon}_2\mathbf{\Omega}'_2$, where $\mathbf{\Upsilon}_2$ $(k \times k)$ is nonsingular, and $\mathbf{\Omega} = (\mathbf{\Omega}_1, \mathbf{\Omega}_2)$ is nonsingular.

To distinguish between the stationary and nonstationary coordinates, we make a transformation of coordinates. Define

$$\mathbf{\Omega}'\mathbf{Y}_t = \mathbf{X}_t = \begin{bmatrix}\mathbf{X}_{1t}\\ \mathbf{X}_{2t}\end{bmatrix}, \quad \mathbf{\Omega}'\mathbf{Z}_t = \mathbf{W}_t = \begin{bmatrix}\mathbf{W}_{1t}\\ \mathbf{W}_{2t}\end{bmatrix},$$

$\mathbf{\Psi}_j = \mathbf{\Omega}'\mathbf{B}_j(\mathbf{\Omega}')^{-1}$, $j = 1, \ldots, m$. Then the process **1** is transformed to

$$\mathbf{X}_t = \mathbf{\Psi}_1\mathbf{X}_{t-1} + \cdots + \mathbf{\Psi}_m\mathbf{X}_{t-m} + \mathbf{W}_t. \qquad \textbf{[5]}$$

If we define $\mathbf{\Upsilon} = \mathbf{\Psi}_1 + \cdots + \mathbf{\Psi}_m - \mathbf{I} = \mathbf{\Omega}'\mathbf{\Pi}(\mathbf{\Omega}')^{-1}$, $\mathbf{\Upsilon}_j = -\Sigma_{i=j+1}^{m}\,\mathbf{\Psi}_j = \mathbf{\Omega}'\mathbf{\Pi}_j(\mathbf{\Omega}')^{-1}$, $\bar{\mathbf{\Upsilon}} = (\mathbf{\Upsilon}_1, \ldots, \mathbf{\Upsilon}_{m-1})$, and $\bar{\Delta}\mathbf{X}_{t-1} = (\Delta\mathbf{X}'_{t-1}, \ldots, \Delta\mathbf{X}'_{t-m+1})'$, the form **2** is transformed to

$$\Delta\mathbf{X}_t = \mathbf{\Upsilon}\mathbf{X}_{t-1} + \bar{\mathbf{\Upsilon}}\bar{\Delta}\mathbf{X}_{t-1} + \mathbf{W}_t. \qquad \textbf{[6]}$$

Note that $\mathbf{\Upsilon} = \mathrm{diag}(\mathbf{0}, \mathbf{\Upsilon}^{22})$.

Define $\Delta\hat{\mathbf{X}}_t^{+}$, $\hat{\mathbf{X}}_{t-1}^{+}$, $\mathbf{S}_{\bar{\Delta}X,\bar{\Delta}X}$, $\mathbf{S}_{\Delta X,\bar{\Delta}X}$, $\mathbf{S}_{\bar{X},\bar{\Delta}X}$, $\hat{\mathbf{S}}^{+}_{\Delta X,\Delta X}$, $\hat{\mathbf{S}}^{+}_{\Delta X,\bar{X}}$, and $\hat{\mathbf{S}}^{+}_{\bar{X},\bar{X}}$ in a manner analogous to the definitions in the Y-coordinates. The reduced rank regression estimator of $\mathbf{\Upsilon}$ is based on the canonical correlations and canonical variates between $\Delta\hat{\mathbf{X}}_t^{+}$ and $\hat{\mathbf{X}}_{t-1}^{+}$ defined by

$$|\hat{\mathbf{S}}^{+}_{\bar{X},\Delta X}\hat{\mathbf{S}}^{+\,-1}_{\Delta X,\Delta X}\hat{\mathbf{S}}^{+}_{\Delta X,\bar{X}} - r^2\hat{\mathbf{S}}^{+}_{\bar{X},\bar{X}}| = 0, \qquad \textbf{[7]}$$

$$\hat{\mathbf{S}}^{+}_{\bar{X},\Delta X}\hat{\mathbf{S}}^{+\,-1}_{\Delta X,\Delta X}\hat{\mathbf{S}}^{+}_{\Delta X,\bar{X}}\mathbf{g} = r^2\hat{\mathbf{S}}^{+}_{\bar{X}\bar{X}}\mathbf{g}, \quad \mathbf{g}'\hat{\mathbf{S}}^{+}_{\bar{X}\bar{X}}\mathbf{g} = 1. \qquad \textbf{[8]}$$

The estimator of $\mathbf{\Upsilon}$ of rank k is $\hat{\mathbf{\Upsilon}}_{(k)} = \hat{\mathbf{S}}^{+}_{\Delta X,\bar{X}}\mathbf{G}_2\mathbf{G}'_2$, where $\mathbf{G}_2 = (\mathbf{g}_{n+1}, \ldots, \mathbf{g}_p)$ and $\mathbf{g}_i$ is the solution for $\mathbf{g}$ in **8** when $r = r_i$, the solution to **7** and $r_1 < \cdots < r_p$. The rest of this paper is devoted to finding the asymptotic distribution of $\{\mathbf{g}_i, r_i\}$. Note that $\hat{\mathbf{\Upsilon}}_{(k)} = \mathbf{\Omega}'\hat{\mathbf{\Pi}}_{(k)}(\mathbf{\Omega}')^{-1}$.

The vectors $\Delta\hat{\mathbf{X}}_t^{+} = \Delta\mathbf{X}_t - \mathbf{S}_{\Delta X,\bar{\Delta}X}\mathbf{S}^{-1}_{\bar{\Delta}X,\bar{\Delta}X}\bar{\Delta}\mathbf{X}_{t-1}$ and $\hat{\mathbf{X}}_{t-1}^{+} = \mathbf{X}_{t-1} - \mathbf{S}_{\bar{X},\bar{\Delta}X}\mathbf{S}^{-1}_{\bar{\Delta}X,\bar{\Delta}X}\bar{\Delta}\mathbf{X}_{t-1}$ are the residuals of $\Delta\mathbf{X}_t$ and $\mathbf{X}_{t-1}$ regressed on $\bar{\Delta}\mathbf{X}_{t-1}$, and r_1 is the maximum correlation between $\Delta\hat{\mathbf{X}}_t^{+}$ and $\hat{\mathbf{X}}_{t-1}^{+}$, which is the correlation between $\Delta\mathbf{X}_t$ and $\mathbf{X}_{t-1}$ after taking account of the dependence "explained" by $\bar{\Delta}\mathbf{X}_{t-1}$. The canonical correlations are the canonical correlations between $(\Delta\mathbf{X}'_t, \bar{\Delta}\mathbf{X}_{t-1})$ and $(\mathbf{X}_{t-1}, \bar{\Delta}\mathbf{X}_{t-1})$ other than ± 1.

The Process

The process $\{\mathbf{X}_t\}$ defined by **5** can be put in the form of the Markov model

$$\begin{bmatrix}\mathbf{X}_t\\ \mathbf{X}_{t-1}\\ \vdots\\ \mathbf{X}_{t-m+1}\end{bmatrix} = \begin{bmatrix}\mathbf{\Psi}_1 & \mathbf{\Psi}_2 & \ldots & \mathbf{\Psi}_{m-1} & \mathbf{\Psi}_m\\ \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_{t-1}\\ \mathbf{X}_{t-2}\\ \vdots\\ \mathbf{X}_{t-m}\end{bmatrix} + \begin{bmatrix}\mathbf{W}_t\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix} \qquad \textbf{[9]}$$

(section 5.4, ref. 10). Multiplication of **9** on the left by

$$\begin{bmatrix}\mathbf{I}_p & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}_{p(m-1)}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{I}_{p(m-1)} & \mathbf{0}\end{bmatrix}$$

yields a form that includes the error-correction form **6**

$$\begin{bmatrix}\mathbf{X}_t\\ \Delta\mathbf{X}_t\\ \Delta\mathbf{X}_{t-1}\\ \vdots\\ \Delta\mathbf{X}_{t-m+2}\end{bmatrix} = \begin{bmatrix}\mathbf{\Upsilon}+\mathbf{I} & \mathbf{\Upsilon}_1 & \mathbf{\Upsilon}_2 & \ldots & \mathbf{\Upsilon}_{m-2} & \mathbf{\Upsilon}_{m-1}\\ \mathbf{\Upsilon} & \mathbf{\Upsilon}_1 & \mathbf{\Upsilon}_2 & \ldots & \mathbf{\Upsilon}_{m-2} & \mathbf{\Upsilon}_{m-1}\\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{0}\end{bmatrix}$$

$$\cdot\begin{bmatrix}\mathbf{X}_{t-1}\\ \Delta\mathbf{X}_{t-1}\\ \Delta\mathbf{X}_{t-2}\\ \vdots\\ \Delta\mathbf{X}_{t-m+1}\end{bmatrix} + \begin{bmatrix}\mathbf{W}_t\\ \mathbf{W}_t\\ \mathbf{0}\\ \vdots\\ \mathbf{0}\end{bmatrix}. \qquad \textbf{[10]}$$

The first n components of **10** constitute

$$\mathbf{X}_{1t} - \mathbf{X}_{1,t-1} = \sum_{j=1}^{m-1}[\mathbf{\Upsilon}_j^{11}(\mathbf{X}_{1,t-j} - \mathbf{X}_{1,t-j-1}) + \mathbf{\Upsilon}_j^{12}(\mathbf{X}_{2,t-j} - \mathbf{X}_{2,t-j-1})] + \mathbf{W}_{1t}. \qquad \textbf{[11]}$$

Here $\mathbf{\Upsilon}_j$ has been partitioned into n and k rows and columns. Assume $\mathbf{X}_{10} = \mathbf{X}_{1,-1} = \ldots = \mathbf{0}$ and $\mathbf{W}_{10} = \mathbf{W}_{1,-1} = \ldots = \mathbf{0}$. The sum of **11** for $t = -\infty$ to $t = s$ is $\mathbf{X}_{1s} = \Sigma_{j=1}^{m-1}\,[\mathbf{\Upsilon}_j^{11}\mathbf{X}_{1,s-j} + \mathbf{\Upsilon}_j^{12}\mathbf{X}_{2,s-j}] + \Sigma_{t=1}^{s}\,\mathbf{W}_{1t}$, or

$$\left(\mathbf{I} - \sum_{j=1}^{m-1}\mathbf{\Upsilon}_j^{11}\right)\mathbf{X}_{1s} = \sum_{j=1}^{m-1}\left[\mathbf{\Upsilon}_j^{12}\left(\mathbf{X}_{2,s-1} - \sum_{i=2}^{j}\Delta\mathbf{X}_{2,s-i}\right) - \mathbf{\Upsilon}_j^{11}\sum_{i=1}^{j}\Delta\mathbf{X}_{1,s-i}\right] + \sum_{t=1}^{s}\mathbf{W}_{1t}. \qquad \textbf{[12]}$$

Write **12** as

$$\mathbf{X}_{1s} = \mathbf{\Gamma}\sum_{t=1}^{s}\mathbf{W}_{1t} + \mathbf{H}\tilde{\mathbf{X}}_{s-1}, \qquad \textbf{[13]}$$

where $\mathbf{\Gamma} = (\mathbf{I} - \Sigma_{j=1}^{m-1}\,\mathbf{\Upsilon}_j^{11})^{-1}$, $\mathbf{\Gamma}^{-1}\mathbf{H}$ is a linear combination of $\mathbf{\Upsilon}_1, \ldots, \mathbf{\Upsilon}_{m-1}$, and $\tilde{\mathbf{X}}_s = (\mathbf{X}'_{2s}, \bar{\Delta}\mathbf{X}'_s)'$. [The matrix on the left-hand side of **12** is nonsingular because otherwise there would be a linear combination of the right-hand side identically 0.] The right-hand side of **13** is the sum of a stationary process and a random walk $(\Sigma_{t=1}^{s}\,\mathbf{W}_{1t})$.

The last $pm - n = k + p(m-1)$ components of **10** constitute a stationary process satisfying

$$\tilde{\mathbf{X}}_t = \tilde{\mathbf{\Upsilon}}\tilde{\mathbf{X}}_{t-1} + \tilde{\mathbf{W}}_t, \qquad \textbf{[14]}$$

where $\tilde{\mathbf{X}}_t' = (\mathbf{X}_{2t}', \Delta\bar{\mathbf{X}}_t')$, $\tilde{\mathbf{W}}_t' = (\mathbf{W}_{2t}', \mathbf{W}_t', \mathbf{0})$, and $\tilde{\mathbf{Y}}$ consists of the last $pm - n$ rows and columns of the coefficient matrix in **10**. Note that the first n columns and last $pm - n$ rows of that matrix consist of **0**s. Because the eigenvalues of $\tilde{\mathbf{Y}}$ are less than 1 in absolute value (9), $\tilde{\mathbf{X}}_t = \Sigma_{s=0}^{\infty} \tilde{\mathbf{Y}}^s \tilde{\mathbf{W}}_{t-s}$, $\mathcal{E}\tilde{\mathbf{X}}_t\tilde{\mathbf{X}}_t' = \tilde{\mathbf{\Sigma}} = \Sigma_{s=0}^{\infty} \tilde{\mathbf{Y}}^s \tilde{\mathbf{\Sigma}}_{WW} \tilde{\mathbf{Y}}'^s$, $\mathcal{E}\tilde{\mathbf{X}}_t\tilde{\mathbf{X}}_{t-h}' = \tilde{\mathbf{Y}}^h\tilde{\mathbf{\Sigma}}$. The covariance $\tilde{\mathbf{\Sigma}}$ satisfies

$$\tilde{\mathbf{\Sigma}} = \tilde{\mathbf{Y}}\tilde{\mathbf{\Sigma}}\tilde{\mathbf{Y}}' + \tilde{\mathbf{\Sigma}}_{WW}. \qquad [15]$$

Given $\tilde{\mathbf{Y}}$ and $\tilde{\mathbf{\Sigma}}_{WW}$, **15** can be solved for $\tilde{\mathbf{\Sigma}}$ [Anderson (10), section 5.5]. Further we write **13** as $\mathbf{X}_{1t} = \mathbf{\Gamma}\ \Sigma_{s=0}^{t}\ \mathbf{W}_{1,t-s} + \mathbf{H}\ \Sigma_{s=0}^{\infty}\ \tilde{\mathbf{Y}}^s\tilde{\mathbf{W}}_{t-s}$. Then

$$\begin{aligned}
\mathcal{E}\mathbf{X}_{1t}\mathbf{X}_{1t}' &= t\mathbf{\Gamma}\mathbf{\Sigma}_{WW}^{11}\mathbf{\Gamma}' + \mathbf{\Gamma}\mathbf{\Sigma}_{W\tilde{W}}^{1\cdot}(\mathbf{I} - \tilde{\mathbf{Y}}'^{t+1})(\mathbf{I} - \tilde{\mathbf{Y}}')^{-1}\mathbf{H}' \\
&\quad + \mathbf{H}(\mathbf{I} - \tilde{\mathbf{Y}})^{-1}(\mathbf{I} - \tilde{\mathbf{Y}}^{t+1})\mathbf{\Sigma}_{\tilde{W}W}^{\cdot 1}\mathbf{\Gamma}' \\
&\quad + \mathbf{H}\sum_{s=0}^{\infty}\tilde{\mathbf{Y}}^s\tilde{\mathbf{\Sigma}}_{WW}\tilde{\mathbf{Y}}'^s\mathbf{H}' \\
&\sim t\mathbf{\Gamma}\mathbf{\Sigma}_{WW}^{11}\mathbf{\Gamma}' + \mathbf{\Gamma}\mathbf{\Sigma}_{W\tilde{W}}^{1\cdot}(\mathbf{I} - \tilde{\mathbf{Y}}')^{-1}\mathbf{H}' \\
&\quad + \mathbf{H}(\mathbf{I} - \tilde{\mathbf{Y}})^{-1}\mathbf{\Sigma}_{\tilde{W}W}^{\cdot 1}\mathbf{\Gamma}' + \mathbf{H}\tilde{\mathbf{\Sigma}}\mathbf{H}'
\end{aligned}$$

since $\mathbf{I} - \tilde{\mathbf{Y}}^t \to \mathbf{I}$. Here $\mathcal{E}\mathbf{W}_{1t}\tilde{\mathbf{W}}_t' = \mathbf{\Sigma}_{W\tilde{W}}^{1\cdot}$ is the second set of rows in $\tilde{\mathbf{\Sigma}}_{WW}$. Then $T^{-1}\mathcal{E}\mathbf{S}_{\bar{X}\bar{X}}^{11} = T^{-2}\ \Sigma_{t=1}^{T}\ \mathcal{E}\mathbf{X}_{1t}\mathbf{X}_{1t}' \to 2^{-1}\mathbf{\Gamma}\mathbf{\Sigma}_{WW}^{11}\mathbf{\Gamma}'$ because $\Sigma_{t=1}^{T}\ t = T(T+1)/2$. Further

$$\begin{aligned}
\mathcal{E}\mathbf{X}_{1t}\tilde{\mathbf{X}}_t' &= \mathcal{E}\left(\mathbf{\Gamma}\sum_{s=0}^{t}\mathbf{W}_{1,t-s} + \mathbf{H}\sum_{s=0}^{\infty}\tilde{\mathbf{Y}}^s\tilde{\mathbf{W}}_{t-s}\right)\sum_{r=0}^{\infty}\tilde{\mathbf{W}}_{t-r}'\tilde{\mathbf{Y}}'^r \\
&= \mathbf{\Gamma}\mathbf{\Sigma}_{W\tilde{W}}^{1\cdot}(\mathbf{I} - \tilde{\mathbf{Y}}'^{t+1})(\mathbf{I} - \tilde{\mathbf{Y}}')^{-1} + \mathbf{H}\tilde{\mathbf{\Sigma}} \\
&\to \mathbf{\Gamma}\mathbf{\Sigma}_{W\tilde{W}}^{1\cdot}(\mathbf{I} - \tilde{\mathbf{Y}}')^{-1} + \mathbf{H}\tilde{\mathbf{\Sigma}}.
\end{aligned}$$

Define

$$\begin{aligned}
\Delta\mathbf{X}_t^{+} &= \Delta\mathbf{X}_t - \mathbf{\Sigma}_{\Delta X,\bar{\Delta}X}\mathbf{\Sigma}_{\bar{\Delta}X,\bar{\Delta}X}^{-1}\bar{\Delta}\mathbf{X}_{t-1}, \\
\mathbf{X}_{t-1}^{+} &= \mathbf{X}_{t-1} - \mathbf{\Sigma}_{\bar{X},\bar{\Delta}X}\mathbf{\Sigma}_{\bar{\Delta}X,\bar{\Delta}X}^{-1}\bar{\Delta}\mathbf{X}_{t-1},
\end{aligned} \qquad [16]$$

where $\mathbf{\Sigma}_{\Delta X,\bar{\Delta}X} = \mathcal{E}\Delta\mathbf{X}_t\bar{\Delta}\mathbf{X}_{t-1}'$, $\mathbf{\Sigma}_{\bar{X},\bar{\Delta}X} = \mathcal{E}\mathbf{X}_{t-1}\bar{\Delta}\mathbf{X}_{t-1}'$ depends on t, and $\mathbf{\Sigma}_{\bar{\Delta}X,\bar{\Delta}X} = \mathcal{E}\bar{\Delta}\mathbf{X}_{t-1}\bar{\Delta}\mathbf{X}_{t-1}'$ does not depend on t. Note that $\Delta\mathbf{X}_t^{+}$ and $\mathbf{X}_{t-1}^{+}$ correspond to $\Delta\hat{\mathbf{X}}_t^{+}$ and $\hat{\mathbf{X}}_{t-1}^{+}$ with $\mathbf{S}_{\Delta X,\bar{\Delta}X}$, $\mathbf{S}_{\bar{X},\bar{\Delta}X}$ and $\mathbf{S}_{\bar{\Delta}X,\bar{\Delta}X}$ replaced by $\mathbf{\Sigma}_{\Delta X,\bar{\Delta}X}$, $\mathbf{\Sigma}_{\bar{X},\bar{\Delta}X}$ and $\mathbf{\Sigma}_{\bar{\Delta}X,\bar{\Delta}X}$, respectively. Then **6** can be written as a regression model

$$\Delta\mathbf{X}_t^{+} = \mathbf{Y}\mathbf{X}_{t-1}^{+} + \mathbf{W}_t \qquad [17]$$

with $\mathcal{E}\mathbf{X}_{t-1}^{+}\mathbf{W}_t' = \mathbf{0}$. Note that this model has the form of 2.10 in Anderson (2).

From **16** and **17** we calculate

$$\begin{aligned}
\mathcal{E}\Delta\mathbf{X}_t^{+}\Delta\mathbf{X}_t^{+\prime} &= \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{22}\mathbf{\Sigma}_{\bar{X}\bar{X}}^{+22}\mathbf{Y}^{22\prime}\end{bmatrix} + \mathbf{\Sigma}_{WW} = \mathbf{\Sigma}_{\Delta X,\Delta X}^{+}, \\
\mathcal{E}\Delta\mathbf{X}_t^{+}\mathbf{X}_{t-1}^{+\prime} &= \begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{Y}^{22}\mathbf{\Sigma}_{\bar{X}\bar{X}}^{+21} & \mathbf{Y}^{22}\mathbf{\Sigma}_{\bar{X}\bar{X}}^{+22}\end{bmatrix} = \mathbf{\Sigma}_{\Delta X,\bar{X}}^{+}, \\
\mathcal{E}\mathbf{X}_{2,t-1}^{+}\mathbf{X}_{2,t-1}^{+\prime} &= \mathbf{\Sigma}_{\bar{X}\bar{X}}^{22} - \mathbf{\Sigma}_{\bar{X},\bar{\Delta}X}^{2\cdot}\mathbf{\Sigma}_{\bar{\Delta}X,\bar{\Delta}X}^{-1}\mathbf{\Sigma}_{\bar{\Delta}X,\bar{X}}^{\cdot 2} = \mathbf{\Sigma}_{\bar{X}\bar{X}}^{+22}.
\end{aligned}$$

The process analogs of **7** and **8** are

$$|\mathbf{\Sigma}_{\bar{X},\Delta X}^{+}\mathbf{\Sigma}_{\Delta X,\Delta X}^{+-1}\mathbf{\Sigma}_{\Delta X,\bar{X}}^{+} - \rho^2\mathbf{\Sigma}_{\bar{X}\bar{X}}^{+}| = 0, \qquad [18]$$

$$\mathbf{\Sigma}_{\bar{X},\Delta X}^{+}\mathbf{\Sigma}_{\Delta X,\Delta X}^{+-1}\mathbf{\Sigma}_{\Delta X,\bar{X}}^{+}\boldsymbol{\gamma} = \rho^2\mathbf{\Sigma}_{\bar{X}\bar{X}}^{+}\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma}'\mathbf{\Sigma}_{\bar{X}\bar{X}}^{+}\boldsymbol{\gamma} = 1. \qquad [19]$$

These define the process canonical correlations and variates in the X-coordinates.

Sample Statistics

The canonical correlations and vectors depend on $\hat{\mathbf{S}}_{\Delta X,\Delta X}^{+}$, $\hat{\mathbf{S}}_{\Delta X,\bar{X}}^{+}$, and $\hat{\mathbf{S}}_{\bar{X}\bar{X}}^{+}$, which in turn depend on the submatrices of $\mathbf{S}_{\bar{X}\bar{X}}$, $\mathbf{S}_{\bar{X},\bar{\Delta}X}$, and $\mathbf{S}_{\bar{\Delta}X,\bar{\Delta}X}$ (equivalently $\mathbf{S}_{\bar{X}\bar{X}}^{11}$, $\tilde{\mathbf{S}}_{\bar{X}\tilde{X}}^{1\cdot}$, $\tilde{\mathbf{S}}$). The vector $\tilde{\mathbf{X}}_t$ satisfies the first-order stationary autoregressive model **14**. The sample covariance matrices $\tilde{\mathbf{S}}_{XX}$, $\tilde{\mathbf{S}}_{WX}$, and $\mathbf{S}_{WW}$ are consistent estimators of $\tilde{\mathbf{\Sigma}}$, **0**, and $\mathbf{\Sigma}_{WW}$, and $\tilde{\mathbf{S}}_{XX}^{*} = \sqrt{T}(\tilde{\mathbf{S}}_{XX} - \tilde{\mathbf{\Sigma}})$, $\tilde{\mathbf{S}}_{WX}^{*} = \sqrt{T}\ \tilde{\mathbf{S}}_{WX}$, $\mathbf{S}_{WW}^{*} = \sqrt{T}\ (\mathbf{S}_{WW} - \mathbf{\Sigma}_{WW})$ have a limiting normal distribution with means **0** and covariances that have been given in refs. 2 and 11.

Let $\mathbf{W}(u)$ be the Brownian motion process defined by $T^{-1/2}\ \Sigma_{t=1}^{[Tu]}\ \mathbf{W}_t \xrightarrow{w} \mathbf{W}(u)$. Define $\mathbf{I}_{11}$ by

$$\frac{1}{T^2}\sum_{t=1}^{T}\sum_{r,s=1}^{t}\mathbf{W}_{1s}\mathbf{W}_{1r}' \xrightarrow{d} \int_0^1 \mathbf{W}_1(u)\mathbf{W}_1'(u)du = \mathbf{I}_{11}.$$

See Anderson (2) and theorem B.12 of Johansen (6). Define $\mathbf{J}_{j1}$ by

$$\frac{1}{T}\sum_{t=1}^{T}\mathbf{W}_{jt}\sum_{s=1}^{t-1}\mathbf{W}_{1s}' \xrightarrow{d} \int_0^1 d\mathbf{W}_j(u)\mathbf{W}_1'(u) = \mathbf{J}_{j1}, \quad j = 1, 2.$$

Then $T^{-1}\mathbf{S}_{\bar{X}\bar{X}}^{11} \xrightarrow{d} \mathbf{\Gamma}\mathbf{I}_{11}\mathbf{\Gamma}'$ by **13**, $T^{-1}\tilde{\mathbf{S}}_{XX} \xrightarrow{p} \mathbf{0}$, and the Cauchy–Schwarz inequality.

We shall find the limit in distribution of $\mathbf{S}_{\bar{X}\bar{X}}^{21}$ from the limit of $\mathbf{S}_{\tilde{X}\bar{X}}^{\cdot 1}$ by using equation B.20 of theorem B.13 of Johansen (6). A specialization to the model here is

$$\frac{1}{T}\sum_{t=1}^{T}\tilde{\mathbf{X}}_t\sum_{s=1}^{t-1}\mathbf{W}_{1s}' \xrightarrow{d} (\mathbf{I} - \tilde{\mathbf{Y}})^{-1}\left[\int_0^1 d\tilde{\mathbf{W}}(u)\mathbf{W}_1'(u) + \mathbf{\Sigma}_{\tilde{W}W}^{\cdot 1}\right] - \mathbf{\Sigma}_{\tilde{W}W}^{\cdot 1},$$

where $\tilde{\mathbf{W}}(u) = [\mathbf{W}_2'(u), \mathbf{W}'(u), \mathbf{0}]'$. [In theorem B.13, let $\theta_i = (\mathbf{I}, \mathbf{0})$, $\psi_i = (\mathbf{0}, \tilde{\mathbf{Y}}^i)$, $\varepsilon_t' = (\mathbf{W}_{1t}', \tilde{\mathbf{W}}_t')$, and $\Omega = \mathcal{E}\varepsilon_t\varepsilon_t'$, $V_t = \tilde{\mathbf{X}}_t$.] Then

$$\begin{aligned}
\mathbf{S}_{\tilde{X}\bar{X}}^{\cdot 1} = \frac{1}{T}\sum_{t=1}^{T}\tilde{\mathbf{X}}_t\mathbf{X}_{1t}' &\xrightarrow{d} (\mathbf{I} - \tilde{\mathbf{Y}})^{-1}\left[\int_0^1 d\tilde{\mathbf{W}}(u)\mathbf{W}_1'(u) + \mathbf{\Sigma}_{\tilde{W}W}^{\cdot 1}\right]\mathbf{\Gamma}' \\
&\quad + \tilde{\mathbf{\Sigma}}\mathbf{H}' = (\mathbf{I} - \tilde{\mathbf{Y}})^{-1}[(\mathbf{J}_{21}', \mathbf{J}_{11}', \mathbf{J}_{21}', \mathbf{0})' + \mathbf{\Sigma}_{\tilde{W}W}^{\cdot 1}]\mathbf{\Gamma}' + \tilde{\mathbf{\Sigma}}\mathbf{H}'.
\end{aligned}$$

Because $\{\tilde{\mathbf{X}}_t\}$ is stationary, $T^{-1}\ \Sigma_{t=1}^{T}\ \mathbf{W}_t\tilde{\mathbf{X}}_{t-1}' \xrightarrow{p} \mathbf{0}$ and

$$\mathbf{S}_{W\bar{X}} = \begin{bmatrix}\mathbf{S}_{W\bar{X}}^{11} & \mathbf{S}_{W\bar{X}}^{12} \\ \mathbf{S}_{W\bar{X}}^{21} & \mathbf{S}_{W\bar{X}}^{22}\end{bmatrix} \xrightarrow{d} \begin{bmatrix}\mathbf{J}_{11}\mathbf{\Gamma}' & \mathbf{0} \\ \mathbf{J}_{21}\mathbf{\Gamma}' & \mathbf{0}\end{bmatrix}. \qquad [20]$$

Now we wish to show that $\Delta\mathbf{X}_t^{+}$ and $\mathbf{X}_{t-1}^{+}$ lead to the same asymptotic results as $\Delta\hat{\mathbf{X}}_t^{+}$ and $\hat{\mathbf{X}}_{t-1}^{+}$. First note that $T^{-1}\mathbf{S}_{\bar{X}\bar{X}}^{11} \xrightarrow{d} \mathbf{\Gamma}\mathbf{I}_{11}\mathbf{\Gamma}'$ and T^{-1} times any other sample covariance converges in probability to **0**. Hence $T^{-1}\mathbf{S}_{\bar{X}\bar{X}}^{+11} \xrightarrow{d} \mathbf{\Gamma}\mathbf{I}_{11}\mathbf{\Gamma}'$ and $T^{-1}\hat{\mathbf{S}}_{\bar{X}\bar{X}}^{+11} \xrightarrow{d} \mathbf{\Gamma}\mathbf{I}_{11}\mathbf{\Gamma}'$. Because $\{\tilde{\mathbf{X}}_t\}$ is stationary, $\{\mathbf{X}_{2t}^{+}\}$ is stationary, and $\mathbf{S}_{\bar{X}\bar{X}}^{+22} \xrightarrow{p} \mathbf{\Sigma}_{\bar{X}\bar{X}}^{+22}$, $\hat{\mathbf{S}}_{\bar{X}\bar{X}}^{+22} \xrightarrow{p} \mathbf{\Sigma}_{\bar{X}\bar{X}}^{+22}$. Moreover $\tilde{\mathbf{S}}_{\bar{X}\bar{X}}^{*} = \sqrt{T}(\tilde{\mathbf{S}}_{\bar{X}\bar{X}} - \tilde{\mathbf{\Sigma}})$ has a limiting normal distribution. Expansion of $\mathbf{S}_{\bar{X}\bar{X}}^{+22}$ and $\hat{\mathbf{S}}_{\bar{X}\bar{X}}^{+22}$ in terms of the submatrices of $\tilde{\mathbf{S}}_{\bar{X}\bar{X}}^{*}$ shows that $\mathbf{S}_{\bar{X}\bar{X}}^{+*22}$ and $\hat{\mathbf{S}}_{\bar{X}\bar{X}}^{+*22}$ have the same limiting normal distribution. (See *Asymptotic Distribution of the Larger Roots*.) Finally, $T^{-1/2}\mathbf{S}_{\bar{X}\bar{X}}^{+21} \xrightarrow{p} \mathbf{0}$ and $T^{-1/2}\hat{\mathbf{S}}_{\bar{X}\bar{X}}^{+21} \xrightarrow{p} \mathbf{0}$ because $\mathbf{S}_{\bar{X}\bar{X}}^{21}$, $\mathbf{S}_{\bar{X},\bar{\Delta}X}^{2\cdot}$, $\mathbf{S}_{\bar{\Delta}X,\bar{\Delta}X}$, and $\mathbf{S}_{\bar{\Delta}X,\bar{X}}^{\cdot 1}$ and hence $\mathbf{S}_{\bar{X}\bar{X}}^{+21}$ have finite limits in distribution.

From **17** we find that $\text{plim}_{T\to\infty}\mathbf{S}_{\Delta X,\Delta X}^{+} = \text{plim}_{T\to\infty}\hat{\mathbf{S}}_{\Delta X,\Delta X}^{+} = \mathbf{\Sigma}_{\Delta X,\Delta X}^{+}$ and

$$\mathbf{S}^{+}_{\Delta X,\bar{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Y}^{22}\mathbf{S}^{+21}_{\bar{X}\bar{X}} & \mathbf{Y}^{22}\mathbf{S}^{+22}_{\bar{X}\bar{X}} \end{bmatrix} + \mathbf{S}^{+}_{W\bar{X}},$$

where $\mathbf{S}^{+}_{W\bar{X}} = \mathbf{S}_{W\bar{X}} - \mathbf{S}_{W,\bar{\Delta}X}\boldsymbol{\Sigma}^{-1}_{\bar{\Delta}X,\bar{\Delta}X}\boldsymbol{\Sigma}_{\bar{\Delta}X,\bar{X}}$, which converges in distribution to the right-hand side of **20**.

As noted above, $\mathbf{S}^{+21}_{\bar{X}\bar{X}} \xrightarrow{d} \mathbf{S}^{+21}_{\infty}$, which consists of the first k rows of the weak limit of $\mathbf{S}^{1}_{\bar{X}\bar{X}}$. Then

$$\mathbf{S}^{+}_{\Delta X,\bar{X}} \xrightarrow{d} \begin{bmatrix} \mathbf{J}_{11}\boldsymbol{\Gamma}' & \mathbf{0} \\ \mathbf{Y}^{22}\mathbf{S}^{+21}_{\infty} + \mathbf{J}_{21}\boldsymbol{\Gamma}' & \mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}} \end{bmatrix}.$$

Asymptotic Distribution of the Smaller Roots

Let $\mathbf{Q}^{+} = \mathbf{S}^{+}_{\bar{X},\Delta X}\,\mathbf{S}^{+^{-1}}_{\Delta X,\Delta X}\,\mathbf{S}^{+}_{\Delta X,\Delta\bar{X}}$. The n smaller roots of $|\mathbf{Q}^{+} - r^{2}\mathbf{S}^{+}_{\bar{X}\bar{X}}| = 0$ converge in probability to 0 and the k larger roots converge to the roots of

$$|(\boldsymbol{\Sigma}^{+}_{\bar{X},\Delta X}\boldsymbol{\Sigma}^{+^{-1}}_{\Delta X,\Delta X}\boldsymbol{\Sigma}^{+}_{\Delta X,\bar{X}})_{22} - r^{2}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}| = 0$$

by the analysis of ref. 2. Let $d = Tr^{2}$. The n smaller roots of $|\mathbf{Q}^{+}_{22} - T^{-1}\,d\mathbf{S}^{+}_{\bar{X}\bar{X}}| = 0$ converge in distribution to the roots of

$$0 = \left|\boldsymbol{\Gamma}\mathbf{J}'_{11}(\boldsymbol{\Sigma}^{11}_{WW})^{-1}\mathbf{J}_{11}\boldsymbol{\Gamma}' - d\boldsymbol{\Gamma}\mathbf{I}_{11}\boldsymbol{\Gamma}'\right| |\boldsymbol{\Gamma}|^{2}\left|\mathbf{J}'_{11}(\boldsymbol{\Sigma}^{11}_{WW})^{-1}\mathbf{J}_{11} - d\mathbf{I}_{11}\right|$$

by the algebra used in ref. 2, section 5, resulting in the same distribution as in ref. 2. The likelihood ratio criterion for testing that the rank of $\mathbf{Y}$ is k (2) is $-2\log\lambda = -T\sum_{i=1}^{n}(1 - r_i^{2}) = \sum_{i=1}^{n} d_i + o_p(1)$, the limiting distribution of which was found by Johansen (4, 6) and was given in equation 5.4 in ref. 2.

Asymptotic Distribution of the Larger Roots

We now turn to deriving the asymptotic distribution of the k larger roots of $|\mathbf{Q}^{+} - r^{2}\mathbf{S}^{+}_{\bar{X}\bar{X}}| = 0$ and the associated vectors solving $\mathbf{Q}^{+}\mathbf{g} = r^{2}\mathbf{S}^{+}_{\bar{X}\bar{X}}\mathbf{g}$. First we show that the asymptotic distribution of $r^{2}_{n+1}, \ldots, r^{2}_{p}$ is the same as the asymptotic distribution of the zeros of $|\mathbf{Q}^{+}_{22} - r^{2}\mathbf{S}^{+22}_{\bar{X}\bar{X}}|$. Then we transform from the X-coordinates to the coordinates of the process canonical correlations and vectors.

Let $\hat{\mathbf{R}}^{2}_{2} = \mathrm{diag}(r^{2}_{n+1}, \ldots, r^{2}_{p})$ and $\mathbf{G}'_{2} = (\mathbf{G}'_{12}, \mathbf{G}'_{22})'$ consist of the corresponding solutions to $\mathbf{Q}^{+}\mathbf{G}_{2} = \mathbf{S}^{+}_{\bar{X}\bar{X}}\mathbf{G}_{2}\hat{\mathbf{R}}^{2}_{2}$. The normalization of the columns of $\mathbf{G}_{2}$ is $\mathbf{G}'_{2}\mathbf{S}^{+}_{\bar{X}\bar{X}}\mathbf{G}_{2} = \mathbf{I}$, that is,

$$\mathbf{I} = (\sqrt{T}\mathbf{G}'_{12}, \mathbf{G}'_{22})\begin{bmatrix} \frac{1}{T}\mathbf{S}^{+11}_{\bar{X}\bar{X}} & \frac{1}{\sqrt{T}}\mathbf{S}^{+12}_{\bar{X}\bar{X}} \\ \frac{1}{\sqrt{T}}\mathbf{S}^{+21}_{\bar{X}\bar{X}} & \mathbf{S}^{+22}_{\bar{X}\bar{X}} \end{bmatrix}\begin{bmatrix} \sqrt{T}\mathbf{G}_{12} \\ \mathbf{G}_{22} \end{bmatrix}. \qquad [21]$$

The probability limit of **21** shows that $\sqrt{T}\mathbf{G}_{12} = O_p(1)$ and $\mathbf{G}_{22} = O_p(1)$. The submatrix equations in $\mathbf{Q}^{+}\mathbf{G}_{2} = \mathbf{S}^{+}_{\bar{X}\bar{X}}\mathbf{G}_{2}\hat{\mathbf{R}}^{2}_{2}$ can be written as

$$\frac{1}{T}\mathbf{Q}^{+}_{11}\sqrt{T}\mathbf{G}_{12} + \frac{1}{\sqrt{T}}\mathbf{Q}^{+}_{12}\mathbf{G}_{22} = \left(\frac{1}{T}\mathbf{S}^{+11}_{\bar{X}\bar{X}}\sqrt{T}\mathbf{G}_{12} + \frac{1}{\sqrt{T}}\mathbf{S}^{+12}_{\bar{X}\bar{X}}\mathbf{G}_{22}\right)\hat{\mathbf{R}}^{2}_{2}, \qquad [22]$$

$$\frac{1}{\sqrt{T}}\mathbf{Q}^{+}_{21}\sqrt{T}\mathbf{G}_{12} + \mathbf{Q}^{+}_{22}\mathbf{G}_{22} = \left(\frac{1}{\sqrt{T}}\mathbf{S}^{+21}_{\bar{X}\bar{X}}\sqrt{T}\mathbf{G}_{12} + \mathbf{S}^{+22}_{\bar{X}\bar{X}}\mathbf{G}_{22}\right)\hat{\mathbf{R}}^{2}_{2}. \qquad [23]$$

Because $T^{-1}\mathbf{Q}^{+}_{11} \xrightarrow{p} \mathbf{0}$, $T^{-1/2}\mathbf{Q}^{+}_{12} \xrightarrow{p} \mathbf{0}$, $T^{-1/2}\mathbf{S}^{+12}_{\bar{X}\bar{X}} \xrightarrow{p} \mathbf{0}$, $T^{-1}\mathbf{S}^{+11}_{\bar{X}\bar{X}} \xrightarrow{d} \boldsymbol{\Gamma}\mathbf{I}_{11}\boldsymbol{\Gamma}'$ and $\hat{\mathbf{R}}^{2}_{2} \xrightarrow{p} \mathbf{R}^{2}_{2} = \mathrm{diag}(\rho^{2}_{n+1}, \ldots, \rho^{2}_{p})$, the probability limit of the left-hand side of **22** is $\mathbf{0}$; this shows that $\sqrt{T}\mathbf{G}_{12} \xrightarrow{p} \mathbf{0}$. Then the asymptotic distribution of $\mathbf{G}_{22}$ is the asymptotic distribution of $\mathbf{G}_{22}$ defined by

$$\mathbf{Q}^{+}_{22}\mathbf{G}_{22} = \mathbf{S}^{+22}_{\bar{X}\bar{X}}\mathbf{G}_{22}\hat{\mathbf{R}}^{2}_{2}, \quad \mathbf{G}'_{22}\mathbf{S}^{+22}_{\bar{X}\bar{X}}\mathbf{G}_{22} = \mathbf{I}, \qquad [24]$$

where the elements of $\hat{\mathbf{R}}^{2}_{2}$ are defined by $|\mathbf{Q}^{+}_{22} - r^{2}\mathbf{S}^{+22}_{\bar{X}\bar{X}}| = 0$. Note that when $\sqrt{T}\mathbf{G}_{12} \xrightarrow{p} \mathbf{0}$ is combined with **23**, we obtain $\mathbf{Q}^{+}_{22}\mathbf{G}_{22} = \mathbf{S}^{+22}_{\bar{X}\bar{X}}\mathbf{G}_{22}\hat{\mathbf{R}}^{2}_{2} + o_p(T^{-1/2})$.

We proceed to find the asymptotic distribution of $\mathbf{G}_{22}$ and $\hat{\mathbf{R}}^{2}_{2}$ defined by **24** in the manner of ref. 2. Let

$$\mathbf{S}^{+*}_{\Delta X,\bar{X}} = \sqrt{T}(\mathbf{S}^{+}_{\Delta X,\bar{X}} - \boldsymbol{\Sigma}^{+}_{\Delta X,\bar{X}}), \quad \mathbf{S}^{+*22}_{\bar{X}\bar{X}} = \sqrt{T}(\mathbf{S}^{+22}_{\bar{X}\bar{X}} - \boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}),$$

$$\mathbf{S}^{+*2,2\cdot 1}_{\bar{X}W} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\mathbf{X}^{+}_{2,t-1}\mathbf{W}'_{2\cdot 1,t},$$

$$\mathbf{S}^{+*2\cdot 1,2\cdot 1}_{WW} = \frac{1}{\sqrt{T}}\left(\sum_{t=1}^{T}\mathbf{W}_{2\cdot 1,t}\mathbf{W}'_{2\cdot 1,t} - \boldsymbol{\Sigma}^{22\cdot 1}_{WW}\right),$$

where $\mathbf{W}_{2\cdot 1,t} = \mathbf{W}_{2t} - \boldsymbol{\Sigma}^{21}_{WW}(\boldsymbol{\Sigma}^{11}_{WW})^{-1}\mathbf{W}_{1t}$ and $\mathcal{E}\mathbf{W}_{2\cdot 1,t}\mathbf{W}'_{2\cdot 1,t} = \boldsymbol{\Sigma}^{22\cdot 1}_{WW} = \boldsymbol{\Sigma}^{22}_{WW} - \boldsymbol{\Sigma}^{21}_{WW}(\boldsymbol{\Sigma}^{11}_{WW})^{-1}\boldsymbol{\Sigma}^{12}_{WW}$. We expand $\sqrt{T}\{\mathbf{Q}^{+}_{22} - [\boldsymbol{\Sigma}^{+}_{\bar{X},\Delta X}(\boldsymbol{\Sigma}^{+}_{\Delta X,\Delta X})^{-1}\boldsymbol{\Sigma}^{+}_{\Delta X,\bar{X}}]_{22}\}$ to obtain

$$\begin{aligned}
&\sqrt{T}\mathbf{Y}^{22}\{\mathbf{Q}^{+}_{22} - [\boldsymbol{\Sigma}^{+}_{\bar{X},\Delta X}(\boldsymbol{\Sigma}^{+}_{\Delta X,\Delta X})^{-1}\boldsymbol{\Sigma}^{+}_{\Delta X,\bar{X}}]_{22}\}\mathbf{Y}^{22\prime} \\
&= -\mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime}\boldsymbol{\Lambda}^{-1}\mathbf{S}^{*2\cdot 1,2\cdot 1}_{WW}\boldsymbol{\Lambda}^{-1}\mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime} \\
&\quad + \mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime}\boldsymbol{\Lambda}^{-1}\mathbf{S}^{+*2\cdot 1,2}_{W\bar{X}}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Sigma}^{22\cdot 1}_{WW} \\
&\quad + \boldsymbol{\Sigma}^{22\cdot 1}_{WW}\boldsymbol{\Lambda}^{-1}\mathbf{Y}^{22}\boldsymbol{\Sigma}^{+*2,2\cdot 1}_{\bar{X}W}\boldsymbol{\Lambda}^{-1}\mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime} \\
&\quad + \mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime}\boldsymbol{\Lambda}^{-1}\mathbf{S}^{+*22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime} + \mathbf{Y}^{22}\mathbf{S}^{+*22}_{\bar{X}\bar{X}}\boldsymbol{\Lambda}^{-1}\mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime} \\
&\quad + \mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime}\boldsymbol{\Lambda}^{-1}\mathbf{Y}^{22}\mathbf{S}^{+*22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime}\boldsymbol{\Lambda}^{-1}\mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime} + o_p(1),
\end{aligned} \qquad [25]$$

where $\boldsymbol{\Lambda} = \mathbf{Y}^{22}\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{Y}^{22\prime} + \boldsymbol{\Sigma}^{22\cdot 1}_{WW}$. See equation 6.5 of ref. 2.

To express the covariances of the sample matrices, we use the "vec" notation. For $\mathbf{A} = (\mathbf{a}_1, \ldots \mathbf{a}_n)$, we define vec $\mathbf{A} = (\mathbf{a}'_1, \ldots, \mathbf{a}'_n)'$. The Kronecker product of two matrices $\mathbf{A} = (\mathbf{a}_{ij})$ and $\mathbf{B}$ is $\mathbf{A} \otimes \mathbf{B} = (\mathbf{a}_{ij}\mathbf{B})$. A basic relation is vec $\mathbf{ABC} = (\mathbf{C}' \otimes \mathbf{A})$ vec $\mathbf{B}$, which implies vec $\mathbf{xy}' =$ vec $\mathbf{x}1\mathbf{y}' = (\mathbf{y} \otimes \mathbf{x})$ vec $1 = \mathbf{y} \otimes \mathbf{x}$. Define the commutator matrix $\mathbf{K}$ as the (square) permutation matrix such that vec $\mathbf{A}' = \mathbf{K}$ vec $\mathbf{A}$ for every square matrix of the same order as $\mathbf{K}$.

Define $\mathbf{C} = (\mathbf{I}, -\boldsymbol{\Sigma}_{2,\bar{\Delta}X}\boldsymbol{\Sigma}^{-1}_{\bar{\Delta}X,\bar{\Delta}X})$ and $\mathbf{D} = [\mathbf{I}, -\boldsymbol{\Sigma}^{21}_{WW}(\boldsymbol{\Sigma}^{11}_{WW})^{-1}, \mathbf{0}]$. Then $\mathbf{X}^{+}_{2t} = \mathbf{C}\tilde{\mathbf{X}}_t$, $\mathbf{W}_{2\cdot 1,t} = \mathbf{D}\tilde{\mathbf{W}}_t$, $\mathbf{D}\tilde{\boldsymbol{\Sigma}}_{WW} = \boldsymbol{\Sigma}^{22\cdot 1}_{WW}\mathbf{J}'_{(k)}$, $\mathbf{C}\tilde{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}}\mathbf{I}'_{(k)}$, $\mathbf{J}'_{(k)} = (\mathbf{I}, \mathbf{0}, \mathbf{I}, \mathbf{0})$, $\mathbf{I}'_{(k)} = (\mathbf{I}, \mathbf{0})$, $\boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}} = \mathbf{C}\tilde{\boldsymbol{\Sigma}}\mathbf{C}'$, and $\boldsymbol{\Sigma}^{22\cdot 1}_{WW} = \mathbf{D}\tilde{\boldsymbol{\Sigma}}_{WW}\mathbf{D}'$.

THEOREM 1. *If the* $\mathbf{W}_t$ *are independently normally distributed,* $\mathbf{S}^{*2\cdot 1,2\cdot 1}_{WW}$, $\mathbf{S}^{+*2\cdot 1,2}_{W\bar{X}}$, *and* $\mathbf{S}^{+*22}_{\bar{X}\bar{X}}$ *have a limiting normal distribution with means* $\mathbf{0}$, $\mathbf{0}$, *and* $\mathbf{0}$ *and covariances*

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}^{*2\cdot 1,2\cdot 1}_{WW}(\mathrm{vec}\,\mathbf{S}^{*2\cdot 1,2\cdot 1}_{WW})' = (\mathbf{I} + \mathbf{K})(\boldsymbol{\Sigma}^{22\cdot 1}_{WW} \otimes \boldsymbol{\Sigma}^{22\cdot 1}_{WW}), \qquad [26]$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}^{+*2\cdot 1,2}_{W\bar{X}}(\mathrm{vec}\,\mathbf{S}^{+*2\cdot 1,2}_{W\bar{X}})' = \boldsymbol{\Sigma}^{+22}_{\bar{X}\bar{X}} \otimes \boldsymbol{\Sigma}^{22\cdot 1}_{WW}, \qquad [27]$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}^{+*2\cdot 1,2}_{W\bar{X}}(\mathrm{vec}\,\mathbf{S}^{+*2\cdot 1,2\cdot 1}_{WW})' = \mathbf{0}, \qquad [28]$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}^{+*22}_{\bar{X}\bar{X}}(\mathrm{vec}\,\mathbf{S}^{*2\cdot 1,2\cdot 1}_{WW})' \to (\mathbf{I} + \mathbf{K})(\mathbf{C} \otimes \mathbf{C})[\mathbf{I} - (\tilde{\mathbf{Y}} \otimes \tilde{\mathbf{Y}})]^{-1}(\tilde{\boldsymbol{\Sigma}}_{WW} \otimes \tilde{\boldsymbol{\Sigma}}_{WW})(\mathbf{D}' \otimes \mathbf{D}'), \qquad [29]$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}^{+*22}_{\bar{X}\bar{X}}(\mathrm{vec}\,\mathbf{S}^{+*2\cdot 1,2}_{W\bar{X}})' \to (\mathbf{I} + \mathbf{K})(\mathbf{C} \otimes \mathbf{C})[\mathbf{I} - (\tilde{\mathbf{Y}} \otimes \tilde{\mathbf{Y}})]^{-1}(\tilde{\mathbf{Y}}\tilde{\boldsymbol{\Sigma}} \otimes \tilde{\boldsymbol{\Sigma}}_{WW})(\mathbf{C}' \otimes \mathbf{D}'), \qquad [30]$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{\bar{X}\bar{X}}^{+*22}(\mathrm{vec}\,\mathbf{S}_{\bar{X}\bar{X}}^{+*22})' = (\mathbf{I}+\mathbf{K})(\mathbf{C}\otimes\mathbf{C})$$

$$\cdot\{[\mathbf{I}-(\tilde{\mathbf{Y}}\otimes\tilde{\mathbf{Y}})]^{-1}(\tilde{\boldsymbol{\Sigma}}\otimes\tilde{\boldsymbol{\Sigma}})$$

$$+(\tilde{\boldsymbol{\Sigma}}\otimes\tilde{\boldsymbol{\Sigma}})[\mathbf{I}-(\tilde{\mathbf{Y}}'\otimes\tilde{\mathbf{Y}}')]^{-1}-(\tilde{\boldsymbol{\Sigma}}\otimes\tilde{\boldsymbol{\Sigma}})\}(\mathbf{C}'\otimes\mathbf{C}'). \quad \mathbf{[31]}$$

LEMMA 1. *If* $\mathbf{X}$ *is normally distributed with* $\mathcal{E}\mathbf{X}=\mathbf{0}$ *and* $\mathcal{E}\mathbf{X}\mathbf{X}' = \boldsymbol{\Sigma}$, *then* $\mathcal{E}\,\mathrm{vec}\,\mathbf{X}\mathbf{X}'(\mathrm{vec}\,\mathbf{X}\mathbf{X}')' = (\mathbf{I}+\mathbf{K})(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}) + \mathrm{vec}\,\boldsymbol{\Sigma}(\mathrm{vec}\,\boldsymbol{\Sigma})'$. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are independent,* $\mathcal{E}\,\mathrm{vec}\,\mathbf{X}\mathbf{X}'(\mathrm{vec}\,\mathbf{Y}\mathbf{Y}')' = \mathrm{vec}\,\mathcal{E}\mathbf{X}\mathbf{X}' \otimes (\mathrm{vec}\,\mathcal{E}'\mathbf{Y}\mathbf{Y}')'$, $\mathcal{E}\,\mathrm{vec}\,\mathbf{X}\mathbf{Y}'(\mathrm{vec}\,\mathbf{X}\mathbf{Y}')' = \mathcal{E}\mathbf{Y}\mathbf{Y}'\otimes\mathcal{E}\mathbf{X}\mathbf{X}'$, and $\mathcal{E}\,\mathrm{vec}\,\mathbf{X}\mathbf{Y}'(\mathrm{vec}(\mathbf{Y}\mathbf{X}')' = \mathbf{K}\mathcal{E}\mathbf{X}\mathbf{X}'\otimes\mathcal{E}\mathbf{Y}\mathbf{Y}'$.

Proof of Theorem 1: First **26** is equivalent to the first expression in *Lemma 1*. Next $\mathrm{vec}\,\mathbf{S}_{W\bar{X}}^{+*2\cdot1,2} = T^{-1/2}\sum_{t=1}^{T}(\mathbf{X}_{2,t-1}^{+}\otimes\mathbf{W}_{2\cdot1,t})$ implies **27** because $\mathbf{X}_{2,t-1}^{+}$ and $\mathbf{W}_{2\cdot1,s}$ are independent for $t-1\le s$. Similarly **28** follows. To prove **29**, **30**, and **31**, we use the following lemma.

LEMMA 2.

$$\mathrm{vec}\,\tilde{\mathbf{S}}_{XX} = [\mathbf{I}-(\tilde{\mathbf{Y}}\otimes\tilde{\mathbf{Y}})]^{-1}[(\mathbf{I}+\mathbf{K})(\tilde{\mathbf{Y}}\otimes\mathbf{I})\mathrm{vec}\,\tilde{\mathbf{S}}_{W\bar{X}} + \mathrm{vec}\,\tilde{\mathbf{S}}_{WW}] + o_p\left(\frac{1}{\sqrt{T}}\right).$$

Proof of Lemma 2: We have from $\tilde{\mathbf{X}}_t = \tilde{\mathbf{Y}}\tilde{\mathbf{X}}_{t-1}+\tilde{\mathbf{W}}_t$

$$\tilde{\mathbf{S}}_{XX} = \tilde{\mathbf{Y}}\tilde{\mathbf{S}}_{\bar{X}\bar{X}}\tilde{\mathbf{Y}}' + \tilde{\mathbf{Y}}\tilde{\mathbf{S}}_{\bar{X}W} + \tilde{\mathbf{S}}_{W\bar{X}}\tilde{\mathbf{Y}}' + \tilde{\mathbf{S}}_{WW}. \quad \mathbf{[32]}$$

Because $\tilde{\mathbf{S}}_{XX} - \tilde{\mathbf{S}}_{\bar{X}\bar{X}} = (1/T)(\tilde{\mathbf{X}}_T\tilde{\mathbf{X}}_T' - \tilde{\mathbf{X}}_0\tilde{\mathbf{X}}_0')$ and $\{\tilde{\mathbf{X}}_t\}$ is a stationary process, $\tilde{\mathbf{S}}_{\bar{X}\bar{X}}$ in **32** can be replaced by $\tilde{\mathbf{S}}_{XX}+o_p(1)$. Then *Lemma 2* results from **32** and $\mathrm{vec}\,\tilde{\mathbf{Y}}\tilde{\mathbf{S}}_{\bar{X}W} = \mathbf{K}\,\mathrm{vec}\,\tilde{\mathbf{S}}_{W\bar{X}}\tilde{\mathbf{Y}}'$. ■

LEMMA 3.

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{W}_{2t}\mathbf{W}_{2t}'(\mathrm{vec}\,\mathbf{W}_{2\cdot1,t}\mathbf{W}_{2\cdot1,t}')'$$

$$= (\mathbf{I}+\mathbf{K})(\boldsymbol{\Sigma}_{WW}^{22\cdot1}\otimes\boldsymbol{\Sigma}_{WW}^{22\cdot1}) + \mathrm{vec}\,\boldsymbol{\Sigma}_{WW}^{22}(\mathrm{vec}\,\boldsymbol{\Sigma}_{WW}^{22\cdot1})'.$$

Proof of Lemma 3: Write $\mathbf{W}_{2t} = \mathbf{W}_{2\cdot1,t} + \boldsymbol{\Sigma}_{WW}^{21}(\boldsymbol{\Sigma}_{WW}^{11})^{-1}\mathbf{W}_{1t}$. Then

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{W}_{2t}\mathbf{W}_{2t}'(\mathrm{vec}\,\mathbf{W}_{2\cdot1,t}\mathbf{W}_{2\cdot1,t}')'$$

$$= \mathcal{E}\{[\mathbf{W}_{2\cdot1,t}+\boldsymbol{\Sigma}_{WW}^{21}(\boldsymbol{\Sigma}_{WW}^{11})^{-1}\mathbf{W}_{1t}]\otimes[\mathbf{W}_{2\cdot1,t}+\boldsymbol{\Sigma}_{WW}^{21}(\boldsymbol{\Sigma}_{WW}^{11})^{-1}\mathbf{W}_{1t}]\}$$

$$\cdot(\mathbf{W}_{2\cdot1,t}\otimes\mathbf{W}_{2\cdot1,t})$$

$$= (\mathbf{I}+\mathbf{K})(\boldsymbol{\Sigma}_{WW}^{22\cdot1}\otimes\boldsymbol{\Sigma}_{WW}^{22\cdot1})$$

$$+\mathrm{vec}\,\boldsymbol{\Sigma}_{WW}^{22\cdot1}(\mathrm{vec}\,\boldsymbol{\Sigma}_{WW}^{22\cdot1})' + \mathrm{vec}[\boldsymbol{\Sigma}_{WW}^{21}(\boldsymbol{\Sigma}_{WW}^{11})^{-1}\boldsymbol{\Sigma}_{WW}^{21}](\mathrm{vec}\,\boldsymbol{\Sigma}_{WW}^{22\cdot1})'$$

from which the lemma follows. ■

Proof of Theorem 1 Continued: Then **29** follows from *Lemma 2*, **26**, and **28**, and **30** follows from *Lemma 2*, **27**, and **28** for $\tilde{\mathbf{X}}_t$. To prove **31**, use *Lemma 2*, **26**, **27**, and **10** to obtain

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{\bar{X}\bar{X}}^{+*22}(\mathrm{vec}\,\mathbf{S}_{\bar{X}\bar{X}}^{+*22})' = (\mathbf{C}\otimes\mathbf{C})[\mathbf{I}-(\tilde{\mathbf{Y}}\otimes\tilde{\mathbf{Y}})]^{-1}$$

$$[(\mathbf{I}+\mathbf{K})(\tilde{\mathbf{Y}}\otimes\mathbf{I})(\tilde{\boldsymbol{\Sigma}}\otimes\tilde{\boldsymbol{\Sigma}}_{WW})(\tilde{\mathbf{Y}}\otimes\mathbf{I})(\mathbf{I}+\mathbf{K})$$

$$+(\mathbf{I}+\mathbf{K})(\tilde{\boldsymbol{\Sigma}}_{WW}\otimes\tilde{\boldsymbol{\Sigma}}_{WW})][\mathbf{I}-(\tilde{\mathbf{Y}}'\otimes\tilde{\mathbf{Y}}')]^{-1}(\mathbf{C}'\otimes\mathbf{C}')$$

$$= (\mathbf{I}+\mathbf{K})(\mathbf{C}\otimes\mathbf{C})[\mathbf{I}-(\tilde{\mathbf{Y}}\otimes\tilde{\mathbf{Y}})]^{-1}[(\mathbf{Y}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{Y}}'\otimes\tilde{\boldsymbol{\Sigma}}_{WW})(\mathbf{I}+\mathbf{K})$$

$$+(\tilde{\boldsymbol{\Sigma}}_{WW}\otimes\tilde{\boldsymbol{\Sigma}}_{WW})][\mathbf{I}-(\tilde{\mathbf{Y}}'\otimes\tilde{\mathbf{Y}}')]^{-1}(\mathbf{C}'\otimes\mathbf{C}'). \quad \mathbf{[33]}$$

Then substitution of $\tilde{\boldsymbol{\Sigma}}_{WW} = \tilde{\boldsymbol{\Sigma}} - \tilde{\mathbf{Y}}\tilde{\boldsymbol{\Sigma}}\tilde{\mathbf{Y}}'$ in **33** yields **31**. ■

Let $\boldsymbol{\Xi}$ be a $k\times k$ matrix such that $\boldsymbol{\Xi}'(\mathbf{Y}^{22}\boldsymbol{\Sigma}_{\bar{X}\bar{X}}^{+22}\mathbf{Y}^{22\prime})\boldsymbol{\Xi} = \boldsymbol{\Theta}$ and $\boldsymbol{\Xi}'\boldsymbol{\Sigma}_{WW}^{22\cdot1}\boldsymbol{\Xi} = \mathbf{I}$, where $\boldsymbol{\Theta} = \mathrm{diag}(\theta_{n+1},\ldots,\theta_p) = \mathbf{R}_2^2(\mathbf{I}-\mathbf{R}_2^2)^{-1}$, $\mathbf{R}_2^2 = \mathrm{diag}(\rho_{n+1}^2,\ldots,\rho_p^2)$, and ρ_i^2 is a root of **18** with $0<\rho_{n+1}^2<\cdots<\rho_p^2$. Let $\mathbf{U}_{2t}^{+} = \boldsymbol{\Xi}'\mathbf{X}_{2t}^{+}$, $\mathbf{V}_{2t} = \boldsymbol{\Xi}'\mathbf{W}_{2t}$, $\mathbf{V}_{1t} = \mathbf{W}_{1t}$, $\boldsymbol{\Delta}_2 = \boldsymbol{\Xi}'(\mathbf{Y}^{22}+\mathbf{I})(\boldsymbol{\Xi}')^{-1}$, $\mathbf{M}_2 = \boldsymbol{\Xi}'\mathbf{Y}^{22}(\boldsymbol{\Xi}')^{-1}$, $\tilde{\boldsymbol{\Xi}} = \mathrm{diag}[\boldsymbol{\Xi}, \mathbf{I}_{m-1}\otimes\mathrm{diag}(\mathbf{I}_n,\boldsymbol{\Xi})]$, $\tilde{\boldsymbol{\Delta}} = \tilde{\boldsymbol{\Xi}}'\tilde{\mathbf{Y}}(\tilde{\boldsymbol{\Xi}}')^{-1}$, $\tilde{\mathbf{U}}_t = \tilde{\boldsymbol{\Xi}}'\tilde{\mathbf{X}}_t$, $\mathbf{C}_U = \boldsymbol{\Xi}'\mathbf{C}(\tilde{\boldsymbol{\Xi}}')^{-1}$. Then $\{\tilde{\mathbf{U}}_t\}$ is generated by $\tilde{\mathbf{U}}_t = \tilde{\boldsymbol{\Delta}}\tilde{\mathbf{U}}_{t-1}+\tilde{\mathbf{V}}_t$, where $\tilde{\mathbf{V}}_t = \tilde{\boldsymbol{\Xi}}'\tilde{\mathbf{W}}_t$ and $\mathbf{U}_{2t}^{+}$ satisfies $\mathbf{U}_{2t}^{+} = \boldsymbol{\Delta}_2\mathbf{U}_{2,t-1}^{+}+\mathbf{V}_{2t}$, $\Delta\mathbf{U}_{2t}^{+} = \mathbf{M}_2\mathbf{U}_{2,t-1}^{+}+\mathbf{V}_{2t}$. Multiplication of **25** on the left by $\boldsymbol{\Xi}'$ and right by $\boldsymbol{\Xi}$ yields

$$\sqrt{T}[\mathbf{M}_2(\mathbf{S}_{\bar{U},\Delta U}^{+}\mathbf{S}_{\Delta U,\Delta U}^{+-1}\mathbf{S}_{\Delta U,\bar{U}}^{+})_{22}\mathbf{M}_2' - \mathbf{R}_2^2\boldsymbol{\Theta}]$$

$$= -\mathbf{R}_2^2\mathbf{S}_{VV}^{*2\cdot1,2\cdot1}\mathbf{R}_2^2 + \mathbf{R}_2^2\mathbf{S}_{V\bar{U}}^{+*2\cdot1,2}\mathbf{M}_2'(\mathbf{I}-\mathbf{R}_2^2)$$

$$+(\mathbf{I}-\mathbf{R}_2^2)\mathbf{M}_2\mathbf{S}_{\bar{U}V}^{+*2,2\cdot1}\mathbf{R}_2^2 + \mathbf{R}_2^2\mathbf{M}_2\mathbf{S}_{\bar{U}\bar{U}}^{+*22}\mathbf{M}_2'$$

$$+\mathbf{R}_2^2\mathbf{M}_2\mathbf{S}_{\bar{U}\bar{U}}^{+*22}\mathbf{M}_2' - \mathbf{R}_2^2\mathbf{M}_2\mathbf{S}_{\bar{U}\bar{U}}^{+*22}\mathbf{M}_2'\mathbf{R}_2^2 + o_p(1). \quad \mathbf{[34]}$$

THEOREM 2. *If the* $\mathbf{V}_t$ *are independently normally distributed,* $\mathbf{S}_{VV}^{*2\cdot1,2\cdot1}$, $\mathbf{S}_{V\bar{U}}^{+*2\cdot1,2}$, *and* $\mathbf{S}_{\bar{U}\bar{U}}^{+*22}$ *have a limiting normal distribution with means* $\mathbf{0}$, $\mathbf{0}$, *and* $\mathbf{0}$ *and covariances*

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{VV}^{*2\cdot1,21}(\mathrm{vec}\,\mathbf{S}_{VV}^{*2\cdot1,2\cdot1})' = (\mathbf{I}+\mathbf{K})(\mathbf{I}\otimes\mathbf{I}),$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{V\bar{U}}^{+*2\cdot1,2}(\mathrm{vec}\,\mathbf{S}_{V\bar{U}}^{+*2\cdot1,2})' = \mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1}\otimes\mathbf{I},$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{V\bar{U}}^{+*2\cdot1,2}(\mathrm{vec}\,\mathbf{S}_{VV}^{*2\cdot1,2\cdot1})' = \mathbf{0},$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{\bar{U}\bar{U}}^{+*22}(\mathrm{vec}\,\mathbf{S}_{VV}^{*2\cdot1,2\cdot1})' = (\mathbf{I}+\mathbf{K})(\mathbf{C}_U\otimes\mathbf{C}_U)$$

$$\cdot[\mathbf{I}-(\tilde{\boldsymbol{\Delta}}\otimes\tilde{\boldsymbol{\Delta}})]^{-1}(\mathbf{J}_{(k)}\otimes\mathbf{J}_{(k)}),$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{\bar{U}\bar{U}}^{+*22}(\mathrm{vec}\,\mathbf{S}_{V\bar{U}}^{*2\cdot1,2})' = (\mathbf{I}+\mathbf{K})(\mathbf{C}_U\otimes\mathbf{C}_U)$$

$$\cdot[\mathbf{I}-(\tilde{\boldsymbol{\Delta}}\otimes\tilde{\boldsymbol{\Delta}})]^{-1}[\tilde{\boldsymbol{\Delta}}\mathbf{I}_{(k)}\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1}\otimes\mathbf{J}_{(k)}],$$

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{\bar{U}\bar{U}}^{+*22}(\mathrm{vec}\,\mathbf{S}_{\bar{U}\bar{U}}^{+*22})' = (\mathbf{I}+\mathbf{K})\{(\mathbf{C}_U\otimes\mathbf{C}_U)$$

$$\cdot[\mathbf{I}-(\tilde{\boldsymbol{\Delta}}\otimes\tilde{\boldsymbol{\Delta}})]^{-1}(\mathbf{I}_{(k)}\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1}\otimes\mathbf{I}_{(k)}\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1})$$

$$+(\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1}\mathbf{I}_{(k)}'\otimes\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1}\mathbf{I}_{(k)})$$

$$\cdot[\mathbf{I}-(\tilde{\boldsymbol{\Delta}}'\otimes\tilde{\boldsymbol{\Delta}}')]^{-1}(\mathbf{C}_U'\otimes\mathbf{C}_U')$$

$$-(\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1}\otimes\mathbf{M}_2^{-1}\boldsymbol{\Theta}\mathbf{M}_2'^{-1})\}. \quad \mathbf{[35]}$$

Let $\mathbf{L}_{2,t-1} = \mathbf{M}_2\mathbf{U}_{2,t-1}(=\boldsymbol{\Xi}'\mathbf{Y}^{22}\mathbf{X}_{2,t-1})$. Then **34** becomes

$$\sqrt{T}[(\mathbf{S}_{\bar{L},\Delta U}^{+}\mathbf{S}_{\Delta U,\Delta U}^{+-1}\mathbf{S}_{\Delta U,\bar{L}}^{+})_{22} - \mathbf{R}^2(\mathbf{I}-\mathbf{R}^2)]$$

$$= -\mathbf{R}_2^2\mathbf{S}_{VV}^{*2\cdot1,2\cdot1}\mathbf{R}_2^2 + \mathbf{R}_2^2\mathbf{S}_{V\bar{L}}^{+*2\cdot1,2}(\mathbf{I}-\mathbf{R}_2^2) + (\mathbf{I}-\mathbf{R}_2^2)\mathbf{S}_{\bar{L}V}^{+*2,2\cdot1}\mathbf{R}_2^2$$

$$+\mathbf{R}_2^2\mathbf{S}_{\bar{L}\bar{L}}^{+*22} + \mathbf{S}_{\bar{L}\bar{L}}^{+*22}\mathbf{R}_2^2 - \mathbf{R}_2^2\mathbf{S}_{\bar{L}\bar{L}}^{+*22}\mathbf{R}_2^2 + o_p(1).$$

The covariances of the limiting normal distribution of $\mathrm{vec}\,\mathbf{S}_{VV}^{*2\cdot1,2\cdot1}$, $\mathrm{vec}\,\mathbf{S}_{V\bar{L}}^{+*2\cdot1,2} = (\mathbf{M}_2\otimes\mathbf{I})\,\mathrm{vec}\,\mathbf{S}_{V\bar{U}}^{+*2\cdot1,2}$, and $\mathrm{vec}\,\mathbf{S}_{\bar{L}\bar{L}}^{+*22} = (\mathbf{M}_2\otimes\mathbf{M}_2)\,\mathrm{vec}\,\mathbf{S}_{\bar{U}\bar{U}}^{+*22}$ are found from *Theorem 2*. We write the transform of **35** as

$$\mathcal{E}\,\mathrm{vec}\,\mathbf{S}_{\bar{L}\bar{L}}^{+*22}(\mathrm{vec}\,\mathbf{S}_{\bar{L}\bar{L}}^{+*22})' = (\mathbf{I}+\mathbf{K})[\boldsymbol{\Phi}^{+}(\boldsymbol{\Theta}\otimes\boldsymbol{\Theta}) + (\boldsymbol{\Theta}\otimes\boldsymbol{\Theta})\boldsymbol{\Phi}^{+\prime} - (\boldsymbol{\Theta}\otimes\boldsymbol{\Theta})], \quad \mathbf{[36]}$$

where

$$\boldsymbol{\Phi}^{+} = (\mathbf{M}_2\mathbf{C}_U\otimes\mathbf{M}_2\mathbf{C}_U)[\mathbf{I}-(\tilde{\boldsymbol{\Delta}}\otimes\tilde{\boldsymbol{\Delta}})]^{-1}(\mathbf{I}_{(k)}\mathbf{M}_2^{-1}\otimes\mathbf{I}_{(k)}\mathbf{M}_2^{-1}). \quad \mathbf{[37]}$$

Let $\mathbf{H}_{22} = (\mathbf{M}_2')^{-1}\boldsymbol{\Xi}^{-1}\mathbf{G}_{22}$ [$= \boldsymbol{\Xi}^{-1}(\mathbf{Y}^{22\prime})^{-1}\mathbf{G}_{22}$]. Then $\mathbf{Q}_{22}^{+}\mathbf{G}_{22} = \mathbf{S}_{\bar{X}\bar{X}}^{+22}\mathbf{G}_{22}\hat{\mathbf{R}}_2^2$ and $\mathbf{G}_{22}'\mathbf{S}_{\bar{X}\bar{X}}^{+22}\mathbf{G}_{22} = \mathbf{I}$ transform to

$$(\mathbf{S}^{+}_{\bar{L},\Delta U}\mathbf{S}^{+-1}_{\Delta U,\Delta U}\mathbf{S}^{+}_{\Delta U,\bar{L}})_{22}\mathbf{H}_{22} = \mathbf{S}^{+22}_{\bar{L}\bar{L}}\mathbf{H}_{22}\hat{\mathbf{R}}_2^2, \quad \mathbf{H}_{22}'\mathbf{S}^{+22}_{\bar{L}\bar{L}}\mathbf{H}_{22} = \mathbf{I}. \qquad [38]$$

Because $(\mathbf{S}^{+}_{\bar{L},\Delta U}\mathbf{S}^{+-1}_{\Delta U,\Delta U}\mathbf{S}^{+}_{\Delta U,\bar{L}})_{22} \xrightarrow{p} \boldsymbol{\Theta}\mathbf{R}_2^2$ and $\mathbf{S}^{+22}_{\bar{L}\bar{L}} \xrightarrow{p} \boldsymbol{\Theta}$, the probability limits of **38** and $h_{ii} > 0$ imply $\mathbf{H}_{22} \xrightarrow{p} \boldsymbol{\Theta}^{-1/2}$.

Define $\mathbf{H}^{*}_{22} = \sqrt{T}(\mathbf{H}_{22} - \boldsymbol{\Theta}^{-1/2})$ and $\hat{\mathbf{R}}_2^{2*} = \sqrt{T}(\hat{\mathbf{R}}_2^2 - \mathbf{R}_2^2)$. Then we can write **38** as

$$\boldsymbol{\Theta}^{-1}\mathbf{P}\boldsymbol{\Theta}^{-1/2} = \boldsymbol{\Theta}^{-1/2}\hat{\mathbf{R}}_2^{2*} + \mathbf{H}^{*}_{22}\mathbf{R}_2^2 - \mathbf{R}_2^2\mathbf{H}^{*}_{22} + o_p(1), \qquad [39]$$

$$\mathbf{H}^{*\prime}_{22}\boldsymbol{\Theta}^{1/2} + \boldsymbol{\Theta}^{1/2}\mathbf{H}^{*}_{22} = -\boldsymbol{\Theta}^{-1/2}\mathbf{S}^{+*22}_{\bar{L}\bar{L}}\boldsymbol{\Theta}^{-1/2} + o_p(1), \qquad [40]$$

where

$$\mathbf{P} = -\mathbf{R}_2^2\mathbf{S}^{*2\cdot1,2\cdot1}_{VV}\mathbf{R}_2^2 + \mathbf{R}_2^2\mathbf{S}^{+*2\cdot1,2}_{V\bar{L}}(\mathbf{I} - \mathbf{R}_2^2),$$

$$+(\mathbf{I} - \mathbf{R}_2^2)\mathbf{S}^{+*2,2\cdot1}_{\bar{L}V}\mathbf{R}_2^2 + \mathbf{R}_2^2\mathbf{S}^{+*22}_{\bar{L}\bar{L}}(\mathbf{I} - \mathbf{R}_2^2).$$

LEMMA 4.

$$\mathcal{E}(\mathbf{I} + \mathbf{K})[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2]\text{vec}\,\mathbf{S}^{+*2\cdot1,1}_{V\bar{L}}$$
$$(\text{vec}\,\mathbf{S}^{+*2\cdot1,2}_{V\bar{L}})'[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2](\mathbf{I} + \mathbf{K})$$
$$= (\mathbf{I} + \mathbf{K})[\mathbf{R}_2^2(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^4](\mathbf{I} + \mathbf{K}). \qquad [41]$$

LEMMA 5.

$$\mathcal{E}[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2]\text{vec}\,\mathbf{S}^{+*22}_{\bar{L}\bar{L}}$$
$$\{-(\text{vec}\,\mathbf{S}^{*2\cdot1,2}_{VV})'(\mathbf{R}_2^2 \otimes \mathbf{R}_2^2) + (\text{vec}\,\mathbf{S}^{+*2\cdot1,1}_{V\bar{L}})'$$
$$[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2](\mathbf{I} + \mathbf{K})\}$$
$$= -[\mathbf{R}_2^2(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^4](\mathbf{I} + \mathbf{K}). \qquad [42]$$

Proof of Lemma 5: We use the facts that $\mathbf{M}_2 = \boldsymbol{\Delta}_2 - \mathbf{I}$, $\mathbf{J}_{(k)}\mathbf{M}_2 = \tilde{\boldsymbol{\Delta}}\mathbf{I}_{(k)} - \mathbf{I}_{(k)} = (\tilde{\boldsymbol{\Delta}} - \mathbf{I})\mathbf{I}_{(k)}$, and $(\mathbf{I} + \mathbf{K})\mathbf{K} = \mathbf{I} + \mathbf{K}$. Then the left-hand side of **42** is

$$[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2](\mathbf{I} + \mathbf{K})(\mathbf{M}_2\mathbf{C}_U \otimes \mathbf{M}_2\mathbf{C}_U)[\mathbf{I} - (\tilde{\boldsymbol{\Delta}} \otimes \tilde{\boldsymbol{\Delta}})]^{-1}$$
$$[(\tilde{\boldsymbol{\Delta}}\mathbf{I}_{(k)} \otimes \mathbf{J}_{(k)}\mathbf{M}_2)(\mathbf{I} + \mathbf{K}) - (\mathbf{J}_{(k)}\mathbf{M}_2 \otimes \mathbf{J}_{(k)}\mathbf{M}_2)]$$
$$\cdot(\mathbf{M}_2^{-1}\mathbf{R}_2^2 \otimes \mathbf{M}_2^{-1}\mathbf{R}_2^2)$$
$$= [(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2](\mathbf{I} + \mathbf{K})(\mathbf{M}_2\mathbf{C}_U \otimes \mathbf{M}_2\mathbf{C}_U)$$
$$\cdot[\mathbf{I} - (\tilde{\boldsymbol{\Delta}} \otimes \tilde{\boldsymbol{\Delta}})]^{-1}\{[\tilde{\boldsymbol{\Delta}} \otimes (\tilde{\boldsymbol{\Delta}} - \mathbf{I})] + [(\tilde{\boldsymbol{\Delta}} - \mathbf{I}) \otimes \tilde{\boldsymbol{\Delta}}]$$
$$- [(\tilde{\boldsymbol{\Delta}} - \mathbf{I}) \otimes (\tilde{\boldsymbol{\Delta}} - \mathbf{I})]\}(\mathbf{I}_{(k)} \otimes \mathbf{I}_{(k)})(\mathbf{M}_2^{-1}\mathbf{R}_2^2 \otimes \mathbf{M}_2^{-1}\mathbf{R}_2^2)$$
$$= -[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2](\mathbf{I} + \mathbf{K})(\mathbf{M}_2\mathbf{C}_U \otimes \mathbf{M}_2\mathbf{C}_U)(\mathbf{I}_{(k)} \otimes \mathbf{I}_{(k)})$$
$$\cdot(\mathbf{M}_2^{-1}\mathbf{R}_2^2 \otimes \mathbf{M}_2^{-1}\mathbf{R}_2^2),$$

which is the right-hand side of **42**. ■

THEOREM 3. *If* $\mathbf{Z}_t$ *are normally distributed and the roots of* **18** *are distinct,*

$$\mathcal{E}\,\text{vec}\,\mathbf{P}(\text{vec}\,\mathbf{P})' = (\mathbf{R}_2^4 \otimes \mathbf{R}_2^4) - [(\mathbf{I} - \mathbf{R}_2^2)\mathbf{R}_2^2 \otimes \mathbf{R}^4\boldsymbol{\Theta})]$$
$$+ (\mathbf{I} + \mathbf{K})[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2][\boldsymbol{\Phi}^{+}(\boldsymbol{\Theta} \otimes \boldsymbol{\Theta}) + (\boldsymbol{\Theta} \otimes \boldsymbol{\Theta})\boldsymbol{\Phi}^{+\prime}]$$
$$\cdot[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2]. \qquad [43]$$

Proof: Theorem 4 follows from *Theorem 2*, **37**, **41**, **42**, and the transpose of **42** and the fact that $\mathbf{K}(\mathbf{R}_2^4 \otimes \mathbf{R}_2^4) = [(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2]\mathbf{K}(\boldsymbol{\Theta} \otimes \boldsymbol{\Theta})[(\mathbf{I} - \mathbf{R}_2^2) \otimes \mathbf{R}_2^2]$. ■

Note that **43** is equation 6.14 of ref. 2 with $\boldsymbol{\Phi}^{+}$ replacing $\boldsymbol{\Phi}$.

Let $\tilde{\mathbf{E}} = \sum_{i=1}^{k} \varepsilon_i(\varepsilon_i' \otimes \varepsilon_i')$, where ε_i is the k-vector with 1 in the ith position and 0s elsewhere. The matrix $\tilde{\mathbf{E}}$ has 1 in the ith row and i,ith column and 0s elsewhere. Define $\mathbf{r}^{2*} = (r^{2*}_{n+1}, \ldots, r^{2*}_{p})'$. Then

$$\mathbf{r}^{2*} = \tilde{\mathbf{E}}\,\text{vec}\,\boldsymbol{\Theta}^{-1/2}\mathbf{P}\boldsymbol{\Theta}^{-1/2}.$$

The matrix $\tilde{\mathbf{E}}$ has the effect of selecting the i,ith element of $\boldsymbol{\Theta}^{-1/2}\mathbf{P}\boldsymbol{\Theta}^{-1/2}$ and placing it in the ith position of $\mathbf{r}^{2*}$.

THEOREM 4. *If the* $\mathbf{Z}_t$ *vectors are independently normally distributed and the roots of* **18** *are distinct, the limiting distribution of* $\mathbf{r}^{2*}$ *is normal with mean* **0** *and covariance matrix*

$$2[(\mathbf{I} - \mathbf{R}_2^2)^2\tilde{\mathbf{E}}\boldsymbol{\Phi}^{+}\tilde{\mathbf{E}}'\mathbf{R}_2^4 + \mathbf{R}_2^4\tilde{\mathbf{E}}\boldsymbol{\Phi}^{+\prime}\tilde{\mathbf{E}}'(\mathbf{I} - \mathbf{R}_2^2)^2]. \qquad [44]$$

In terms of the components of $\mathbf{r}^{2*}$ the asymptotic covariance of r_i^{2*} and r_j^{2*} is $2[(1 - \rho_i^2)^2\phi^{+}_{ii,jj}\rho_j^4 + \rho_i^4(\phi^{+}_{jj,ii}(1 - \rho_j^2)^2]$. Here $\phi^{+}_{ii,jj}$ denotes the element in the ith row of the ith block of rows and the jth column of the jth block of columns in $\boldsymbol{\Phi}^{+}$.

We now derive the limiting distribution of $\mathbf{H}^{*}_{22} = \mathbf{H}^{*}_{\text{d}} + \mathbf{H}^{*}_{\text{n}}$, where $\mathbf{H}^{*}_{\text{d}} = \text{diag}(h^{*}_{n+1,n+1}, \ldots, h^{*}_{\text{pp}})$. From vec $\mathbf{H}^{*}_{22}\mathbf{R}_2^2 = (\mathbf{R}_2^2 \otimes \mathbf{I})$ vec $\mathbf{H}^{*}_{22}$, and vec $\mathbf{R}_2^2\mathbf{H}^{*}_{22} = (\mathbf{I} \otimes \mathbf{R}_2^2)$ vec $\mathbf{H}^{*}_{22}$ we obtain vec$(\mathbf{H}^{*}_{22}\mathbf{R}_2^2 - \mathbf{R}_2^2\mathbf{H}^{*}_{22}) = \mathbf{N}\mathbf{H}^{*}_{22} = \mathbf{N}\mathbf{H}^{*}_{22}$, where

$$\mathbf{N} = (\mathbf{R}_2^2 \otimes \mathbf{I}) - (\mathbf{I} \otimes \mathbf{R}_2^2)$$
$$= \text{diag}(0, \rho^2_{n+1} - \rho^2_{n+2}, \ldots, \rho^2_{n+1} - \rho^2_{p}, 0, \ldots, \rho^2_{p} - \rho^2_{p-1}, 0).$$

The Moore–Penrose generalized inverse of $\mathbf{N}$ (denoted $\mathbf{N}^{+}$) has a 0 where $\mathbf{N}$ has a 0 and has $(\rho_i^2 - \rho_j^2)^{-1}$ where $\mathbf{N}$ has $(\rho_i^2 - \rho_j^2)$, $i \neq j$. Note that $\mathbf{N}\mathbf{N}^{+} = (\mathbf{I} \otimes \mathbf{I}) - \mathbf{E}$, where $\mathbf{E} = \sum_{i=1}^{k}(\varepsilon_i \otimes \varepsilon_i)(\varepsilon_i' \otimes \varepsilon_i')$. The $k^2 \times k^2$ matrix $\mathbf{E}$ is idempotent of rank k; the $k^2 \times k^2$ matrix $\mathbf{N}\mathbf{N}^{+}$ is idempotent of rank $k^2 - k$; and $\mathbf{E}$ is orthogonal to $\mathbf{N}$ and $\mathbf{N}^{+}$.

From **39** we obtain vec $\mathbf{H}^{*}_{\text{n}} = \mathbf{N}^{+}(\boldsymbol{\Theta}^{-1/2} \otimes \boldsymbol{\Theta}^{-1})$vec $\mathbf{P}$. From **40** we find $\mathbf{H}^{*}_{\text{d}} = -\frac{1}{2}\boldsymbol{\Theta}^{-3/2}\,\text{diag}\mathbf{S}^{+*22}_{\bar{L}\bar{L}}$ and vec $\mathbf{H}^{*}_{\text{d}} = -\frac{1}{2}\,\mathbf{E}\boldsymbol{\Theta}^{-3/2}\,\text{vec}\,\mathbf{S}^{+*22}_{\bar{L}\bar{L}}$.

THEOREM 5. *If the* $\mathbf{Z}_t$ *vectors are independently normally distributed and the roots of* **18** are distinct, vec $\mathbf{H}^{*}_{\text{n}}$ *and* vec $\mathbf{H}^{*}_{\text{d}}$ *have a limiting normal distribution with means* **0** *and* **0** *and covariances*

$$\mathbf{N}^{+}[(\mathbf{I} - \mathbf{R}_2^2) \otimes (\mathbf{I} - \mathbf{R}_2^2)] + \mathbf{N}^{+}[\mathbf{R}_2^{-1}(\mathbf{I} - \mathbf{R}_2^2)^{3/2} \otimes (\mathbf{I} - \mathbf{R}_2^2)]$$
$$\cdot(\mathbf{I} + \mathbf{K})\{\boldsymbol{\Phi}^{+}(\boldsymbol{\Theta} \otimes \boldsymbol{\Theta}) + (\boldsymbol{\Theta} \otimes \boldsymbol{\Theta})\boldsymbol{\Phi}^{+\prime}\}$$
$$\cdot[\mathbf{R}_2^{-1}(\mathbf{I} - \mathbf{R}_2^2)^{3/2} \otimes (\mathbf{I} - \mathbf{R}_2^2)]\mathbf{N}^{+}$$

and

$$\frac{1}{2}\,\mathbf{E}[(\boldsymbol{\Theta}^{-1} \otimes \mathbf{I}) + (\boldsymbol{\theta}^{-3/2} \otimes \mathbf{I})\boldsymbol{\Phi}^{+}(\boldsymbol{\Theta}^{1/2} \otimes \mathbf{I})$$
$$+ (\boldsymbol{\Theta}^{1/2} \otimes \mathbf{I})\boldsymbol{\Phi}^{+\prime}(\boldsymbol{\Theta}^{-3/2} \otimes \mathbf{I})]\mathbf{E},$$

respectively.

From $\mathbf{G}_{22} = \mathbf{Y}^{22\prime}\boldsymbol{\Xi}\mathbf{H}_{22}$ we can transform *Theorem 5* into the asymptotic covariances of vec $\mathbf{G}_{22} = (\mathbf{I} \otimes \mathbf{Y}^{22\prime}\boldsymbol{\Xi})$ vec $\mathbf{H}_{22}$.

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