Generator problem for certain property T factors

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We show that the property T factors associated with groups $SL_n(\mathbf{Z})$, $n \geq 3$ are generated by two selfadjoint elements, one of which has an arbitrarily small support. This answers a question of Dan Voiculescu.

The generator problem for von Neumann algebras is one of the longstanding open questions in Operator Algebras. One of the first results on the problem is due to von Neumann, who showed that abelian von Neumann algebras acting on a separable Hilbert space is generated by a selfadjoint element. Let $\mathcal{H}$ denote a separable Hilbert space (finite or infinite dimensional) and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. It is well known that $\mathcal{B}(\mathcal{H})$ can be generated by a unitary operator and a rank one projection, by two selfadjoint operators (equivalently, one nonselfadjoint element), or by two unitary operators with order 2 and 3, respectively. Similar results hold for a large class of subalgebras of $\mathcal{B}(\mathcal{H})$. For example, all properly infinite von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ have generators with similar properties (see, e.g., ref. 1). Whether all von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$ are generated by two selfadjoint elements remains unsolved. The question has been reduced to a special class of von Neumann algebras, i.e., the factors of type II_1. Partial results were obtained by several authors. Among them is the result by Popa (2), who shows that factors with a Cartan subalgebra are singly generated. More recently, Popa and Ge (3) show that the same is true for property Γ factors. In his recently developed theory of free probability and free entropy, Voiculescu introduces a notion of free entropy dimension. It is believed that free entropy dimension of the generators of a factor of type II_1 is closely related to the minimal number of generators of the factor. After showing that certain property T factors have generators with free entropy dimension less than or equal to one, Voiculescu asks in ref. 4 whether the property T factors associated with groups $SL_n(\mathbf{Z})$, $n \geq 3$ are generated by two selfadjoint elements, one of which can have an arbitrarily small support (in the sense that its range projection has an arbitrarily small trace). In this note, we give a positive answer to this question. Our analysis and results on these property T factors may be helpful in the study of other questions that remain unsolved for this class of factors (for example, Kadison's similarity problem, vanishing higher cohomology groups, etc.).

In the following, we shall introduce some definitions concerning von Neumann algebras and prove that the factors considered in ref. 5 with "cyclic normalizing" generators can in fact be generated by two selfadjoint elements and another selfadjoint element with an arbitrarily small support. By proving some technical results on groups $SL_n(\mathbf{Z})$, we show that their associated group von Neumann algebras can be generated by any Haar unitary operator and another selfadjoint element with an arbitrarily small support. Finally, we introduce a new notion, "generating length," in connection with the generator problem, and obtain a basic result.

Definitions and Basic Properties

von Neumann algebras were introduced by von Neumann in ref. 6. They are strong operator topology closed selfadjoint subalgebras of $\mathcal{B}(\mathcal{H})$, the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. Factors are von Neumann algebras whose center consists of scalar multiples of the identity. von Neumann showed that von Neumann algebras are direct sums (or direct "integrals") of factors. Factors are classified into more types by means of a relative dimension function. Finite factors are those for which this dimension function has a finite range. Otherwise, the factor is called infinite. For the generator question, there is an easy answer to all infinite factors: every infinite factor, or more generally, every properly infinite von Neumann algebra with a separable predual (or acting on a separable Hilbert space), is generated by two selfadjoint elements. A similar result holds for finite dimensional factors, i.e., finite dimensional full matrix algebras. Infinite dimensional finite factors are called factors of type II_1. They arise naturally from regular representations of (discrete) groups.

Suppose G is a countable discrete group with unit e. Let $\mathcal{H}$ be the Hilbert space $l^2(G)$ and L_g the left translation by g^{-1} of functions in $l^2(G)$. Then $g \to L_g$ is a unitary representation of G on $\mathcal{H}$. Let $\mathcal{L}_G$ be the von Neumann algebra generated by $\{L_g : g \in G\}$. In general, $\mathcal{L}_G$ is a finite von Neumann algebra. It is a factor of type II_1 if and only if each conjugacy class in G (other than that of e) is infinite (in this case, G is called an i.c.c. group). The vector state associated with the characteristic function at e (or any other group element) is a tracial state, denoted by τ. In fact, there is one and only one tracial state on each factor of type II_1. We refer to ref. 7 for basics on von Neumann algebras.

In this note, we will be concerned with von Neumann algebras arising from groups $SL_n(\mathbf{Z})$ for $n \geq 3$. [This class of groups have Kazhdan's property T (8). Property T for von Neumann algebras was introduced by Connes and Jones (9).] When n is even, I and $-I$ lie in the center of $SL_n(\mathbf{Z})$. We consider the group $PSL_n(\mathbf{Z})$ ($\cong SL_n(\mathbf{Z})/\{I, -I\}$) in place of $SL_n(\mathbf{Z})$ for the even case. For simplicity of notation, we shall denote $SL_n(\mathbf{Z})$, when n is odd, or $PSL_n(\mathbf{Z})$, when n is even, by G_n. From ref. 10, we know that G_n ($n \geq 3$) is generated by $g_{jk} = I + e_{jk}$ for $j \neq k$ and $1 \leq j, k \leq n$, where I is the identity matrix and each e_{jk} is the matrix unit with (j, k)-entry equal to one and zero elsewhere. [Here G_2 is the subgroup of $SL_2(\mathbf{Z})$ generated by g_{12} and g_{21}. It is used only in the proof of *Lemma 2*.] It is easy to show that each g_{jk} gives rise to a Haar unitary element in $\mathcal{L}_{SL_n(\mathbf{Z})}$ (i.e., the spectral measure of $L_{g_{jk}}$ given by the trace τ is the Haar measure on the unit circle, the spectrum of $L_{g_{jk}}$). It is also easy to verify that g_{jk} commutes with g_{st} whenever $j = s$ or $k = t$.

Now we prove a general result on generators of finite factors.

PROPOSITION 1. *Suppose $\mathcal{M}$ is a factor of type II_1 and is generated by Haar unitary operators $U_1, U_2, \ldots, U_m, \ldots$. Assume that $U_{j+1}^* U_j U_{j+1}$ belongs to the von Neumann subalgebra generated by $U_1, \ldots, U_j$. Then $\mathcal{M}$ is generated by a hyperfinite subfactor and two selfadjoint operators.*

Proof: First we choose a hyperfinite subfactor $\mathcal{R}$ of $\mathcal{M}$ such that $\mathcal{R}' \cap \mathcal{M} = \mathbf{C}I$ (see ref. 11). Let $\mathcal{R}_j$ be the von Neumann subalgebra of $\mathcal{M}$ generated by $\mathcal{R}$ and $U_1, \ldots, U_j$, for $j \geq 1$. Then $\mathcal{R}_j' \cap \mathcal{M} = \mathbf{C}I$ and $\mathcal{R}_j$ is a factor for every j. For any given large integer n, we can find unital embeddings $M_n(\mathbf{C}) \subset M_{n^2}(\mathbf{C}) \subset \cdots \subset \mathcal{R}$. Let $\{E_{jk}^{(i)}\}_{j,k=1}^{n^i}$ be a matrix unit system for $M_{n^i}(\mathbf{C}) \subset \mathcal{R}$, $i = 1, 2, \ldots$. From the assumption that U_i is a Haar unitary element,

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we choose projections $F_1^{(i)}, \ldots, F_{n^i}^{(i)}$ in the abelian von Neumann algebra generated by U_i such that $\tau(F_j^{(i)}) = 1/n^i$ and $\Sigma_{j=1}^{n^i} F_j^{(i)} = I$. Because $\mathcal{R}_i$ (containing U_i) is a factor, we can find a unitary element W_i in $\mathcal{R}_i$ such that $W_i F_j^{(i)} W_i^* = E_{jj}^{(i)}$ for $j = 1, \ldots, n^i$. From our assumption, we have $U_{i+1}^* F_j^{(i)} U_{i+1} \in \mathcal{R}_i$. Again, there is a unitary element V_i in $\mathcal{R}_i$ such that $V_i E_{jj}^{(i)} V_i^* = U_{i+1}^* F_j^{(i)} U_{i+1} = U_{i+1}^* W_i^* E_{jj}^{(i)} W_i U_{i+1}$ for $j = 1, \ldots, n^i$. Thus $W_i U_{i+1} V_i$ commutes with $E_{11}^{(i)}, \ldots, E_{n^i}^{(i)} n^i$ and

$$W_i U_{i+1} V_i = \sum_{j=1}^{n^i} E_{jj}^{(i)} W_i U_{i+1} V_i E_{jj}^{(i)},$$

where $i = 1, 2, \ldots$. Given any i, with respect to the matrix subalgebra $M_{n^i}(\mathbf{C})$ of $\mathcal{R}$, $W_i U_{i+1} V_i$ has at most n^i nonzero diagonal entries. Now we define, inductively, elements T_i in $\mathcal{M}$. When $i = 1$, choose integer m_1 so that $(m_1(m_1 - 1)/2) \geq n$ and $m_1 \leq \sqrt{2n} + 1$. Thus there is an injective map from $\{1, 2, \ldots, n\}$ to the set $\{(j, k) : 1 \leq j \leq k \leq m_1\}$. We shall use $(\alpha_1(j), \beta_1(j))$ to denote the image of j under this map. Let

$$T_1 = \sum_{j=1}^{n} E_{\alpha_1(j)j}^{(1)} W_1 U_2 V_1 E_{j\beta_1(j)}^{(1)}.$$

Then we know that T_1 is strictly upper triangular with respect to matrix units $\{E_{jk}^{(1)}\}_{j,k=1}^{m_1}$ and $W_1 U_2 V_1$ is in the algebra generated by T_1 and $M_n(\mathbf{C})$. The support Q_1 of T_1 is majorized by the sum of the projections $E_{11}^{(1)}, \ldots, E_{m_1}^{(1)} m_1$. Thus $\tau(Q_1) \leq m_1/n \leq (\sqrt{2n} + 1/n)$. Similarly we define T_2 for $W_2 U_3 V_2$ so that its support Q_2 is majorized by the sum of some diagonal projections in $M_{n^2}(\mathbf{C})$, which are orthogonal to Q_1 and $\tau(Q_2) \leq (\sqrt{2n^2} + 1/n^2)$. Continuing this process, we shall have T_i, in a strictly upper triangular form with respect to $M_{n^i}(\mathbf{C})$, such that $W_i U_{i+1} V_i$ lies in the algebra generated by $M_{n^i}(\mathbf{C})$ and T_i. The support Q_i of T_i is orthogonal to $Q_1 + \cdots + Q_{i-1}$ and $\tau(Q_i) \leq (\sqrt{2n^i} + 1/n^i)$. Now let $T = (T_1/\|T_1\|) + (T_2/\|T_2\|) + \cdots$ (or the strong operator limit of $(T_1/\|T_1\|) + \cdots + (T_i/\|T_i\|)$, and $B = T + T^*$. Then it is easy to show that $\mathcal{M}$ is generated by $\mathcal{R}$, U_1, and B. If Q is the support of B, then $\tau(Q) \leq \Sigma_{i=1}^{\infty} (\sqrt{2n^i} + 1/n^i)$, which can be arbitrarily small when n is large. Now $\mathcal{M}$ is generated by $\mathcal{R}$, U_1, and B. Replacing U_1 by a selfadjoint element, we have our desired result. ∎

Remarks: (*i*) The factorial assumption on $\mathcal{M}$ is essential. For example, $\mathcal{L}_{F_m} \bar{\otimes} \mathcal{L}_{\mathbf{Z}}$ satisfies the assumptions of the proposition. But we do not know whether it is generated by four selfadjoint elements (for $m > 4$).

(*ii*) It is well known that the hyperfinite II_1 factor is generated by a small projection and a selfadjoint element. If we choose this projection so that it is orthogonal to the support of B, then $\mathcal{M}$ as given in the proposition is generated by two selfadjoint elements and another with an arbitrarily small support.

Next, we shall show that $\mathcal{L}_{SL_n(\mathbf{Z})}$, $n \geq 3$ is generated by two selfadjoint elements, one of which has an arbitrarily small support.

Generators of $\mathcal{L}_{SL_n(\mathbf{Z})}$ for $n \geq 3$

Recall that $SL_n(\mathbf{Z})$, when n is odd, or $PSL_n(\mathbf{Z})$, when n is even, is denoted by G_n. From the above proposition and *Remark ii*, we see that $\mathcal{L}_{G_n}$ is generated by three selfadjoint elements, one of which has an arbitrarily small support. Now we are going to reduce the number three to two. We choose, again, the generators $\{g_{jk} : 1 \leq j, k \leq n, j \neq k\}$ for G_n defined above. Their corresponding unitary operators are denoted by $L_{g_{jk}}$. We shall embed G_n into G_{n+1} in a natural way. For $g_{jk} = I + e_{jk}$ in G_n, we regard this g_{jk} as an element in G_{n+1} by putting 1 at its $(n + 1, n + 1)$ entry and zeros on the rest of the $(n + 1)$th row and $(n + 1)$th column. It is well known that G_n is an i.c.c. group. Here we prove that many of its subgroups are i.c.c. groups.

LEMMA 2. *If G is a subgroup of G_n, for $n \geq 3$, containing G_{n-1}, then G is an i.c.c. group.*

Proof: We shall show that, for any element g in G, if the conjugacy class of g by the subgroup G_{n-1} is finite, then g is the identity matrix.

Suppose $g = \Sigma_{j,k=1}^{n} a_{jk} e_{jk}$, where $a_{jk} \in \mathbf{Z}$ and $\{e_{jk}\}$ is the matrix unit system. For $g_{st} = I + e_{st}$ in G_{n-1}, $g_{st}^m = I + m e_{st}$ for any m in $\mathbf{Z}$, where $1 \leq s, t \leq n - 1$ and $s \neq t$. Now,

$$\begin{aligned} g_{st}^{-m} g g_{st}^m &= (I - m e_{st}) \left(\sum_{j,k=1}^{n} a_{jk} e_{jk} \right) (I + m e_{st}) \\ &= \sum_{j,k=1}^{n} a_{jk} e_{jk} - m \sum_{j,k=1}^{n} a_{jk} e_{st} e_{jk} + m \sum_{j,k=1}^{n} a_{jk} e_{jk} e_{st} \\ &\quad - m^2 \sum_{j,k=1}^{n} a_{jk} e_{st} e_{jk} e_{st} \\ &= \sum_{j,k=1}^{n} a_{jk} e_{jk} - m \sum_{k=1}^{n} a_{tk} e_{sk} + m \sum_{j=1}^{n} a_{js} e_{jt} \\ &\quad - m^2 a_{ts} e_{st}. \end{aligned}$$

From our assumption that $\{g_{st}^{-m} g g_{st}^m : m \in \mathbf{Z}\}$ is a finite set, we know that $\{m^2 a_{st} e_{st} : m \in \mathbf{Z}\}$ is a finite set. Thus $a_{st} = 0$ for all $1 \leq s, t \leq n - 1$ and $s \neq t$. Similarly, $\Sigma_{k=1}^{n} a_{tk} e_{sk} - \Sigma_{j=1}^{n} a_{js} e_{jt} = 0$. Thus, from $a_{st} = 0$ for $s \neq t$, we have

$$a_{tt} e_{st} + a_{tn} e_{sn} - a_{ss} e_{st} - a_{ns} e_{nt} = 0.$$

This implies that $a_{ss} = a_{tt}$ and $a_{ns} = a_{tn} = 0$. Therefore g must be the unit I in G_n. ∎

The following result is easy to check. We omit its proof here.

LEMMA 3. *The abelian group generated by g_{js} and g_{jt}, two of the generating elements of G_n, is isomorphic to $\mathbf{Z} \times \mathbf{Z}$, i.e., the Haar unitary operators $L_{g_{js}}$ and $L_{g_{jt}}$ are commuting independent elements. The same is true for g_{sj} and g_{tj}.*

Before we state our main theorem, we prove another technical result.

LEMMA 4. *Let $\mathcal{M}$ be a factor of type II_1 with trace τ. Suppose $\mathcal{M}$ is generated by a subfactor $\mathcal{N}$ and a unitary operator U in $\mathcal{M}$, and there is a Haar unitary operator V in $\mathcal{N}$ such that $U^*VU \in \mathcal{N}$. Assume $\mathcal{N}$ is generated by a subalgebra $\mathcal{N}_m$, isomorphic to $M_m(\mathbf{C})$, and a selfadjoint element S* whose support is majorized by a projection *P in $\mathcal{N}_m$ with $\tau(P) \leq \varepsilon$. Then $\mathcal{M}$ is generated by $\mathcal{N}_m$ and a selfadjoint element T whose support is majorized by a projection Q in $\mathcal{N}_m$ such that $\tau(Q) \leq \varepsilon + (\sqrt{2m} + 1/m)$.*

Proof: Because V is a Haar unitary element, there are projections $F_1, \ldots, F_m$ in the abelian von Neumann algebra generated by V such that $\Sigma_{j=1}^{m} F_j = I$ and $\tau(F_j) = 1/m$. Choose $\{E_{jk}\}_{j,k=1}^{m}$ in $\mathcal{N}_m$, which correspond to a matrix unit system in $M_m(\mathbf{C})$ such that P is the sum of diagonal projections $E_{11}, \ldots, E_{m_1 m_1}$ for some m_1 (and $(m_1/m) \leq \varepsilon$). Because $\mathcal{N}$ is a factor, there is a unitary operator W in $\mathcal{N}$ such that $F_j = W^* E_{jj} W$, $j = 1, \ldots, m$. Let $Q_j = U^* F_j U$, for $j = 1, \ldots, m$. Then, from our assumption, we know that $Q_j \in \mathcal{N}$. Again, there is a unitary element W_1 in $\mathcal{N}$ such that $Q_j = W_1^* E_{jj} W_1$. Now we have that $W_1^* E_{jj} W_1 = U^* F_j U = U^* W^* E_{jj} W U$. Thus

$$E_{jj} W U W_1^* = W U W_1^* E_{jj}, \quad j = 1, \ldots, m,$$

i.e., $WUW_1^* = \Sigma_{j=1}^m E_{jj} WUW_1^* E_{jj}$. We use a trick similar to the one we used in the proof of *Proposition 1* and choose m_2 so that $(m_2(m_2 - 1)/2) \geq m$ and $m_2 \leq (\sqrt{2m} + 1/m)$. Consider a matrix subalgebra of $\mathcal{N}_m$ with matrix units E_{jk} *for* $j, k = m_1 + 1, \ldots, m_1 + m_2$. Now there are more than m entries in the upper triangular subalgebra of this matrix subalgebra. We can define an injective map $\sigma : \{1, \ldots, m\} \to \{(j, k) : j < k, j, k = m_1 + 1, \ldots, m_1 + m_2\}$. We write $\sigma(j) = (\sigma_1(j), \sigma_2(j))$. Now let

$$T_1 = \sum_{j=1}^{m} E_{\sigma_1(j)j} WUW_1^* E_{j\sigma_2(j)}$$

and $T' = T_1 + T_1^*$. Then T' is selfadjoint and its support is majorized by $\Sigma_{j=m_1+1}^{m_1+m_2} \mathrm{E_{jj}}$. Let $T = S + \lambda P + T'$. By choosing an appropriate real constant λ, we have that both T' and $S + \lambda P$ lie in the von Neumann algebra generated by T. It is easy to see that $\mathcal{N}_m$ and T generate $\mathcal{M}$. It is easy to check the estimate for the trace of the support of T. This completes the proof. ■

The following is the main result of this article.

THEOREM 5. *The factor* $\mathcal{L}_{G_n}$, *for* $n \geq 3$, *is generated by two selfadjoint elements, one of which can have an arbitrarily small support.*

Proof: We shall use the generators g_{jk}, $1 \leq j, k \leq n$ and $j \neq k$, for the group G_n. Let H be the subgroup of G_n generated by g_{21}, g_{31}, g_{32} and g_{12}. The H is a subgroup of G_3 containing G_2. From *Lemma 2*, we know that $\mathcal{L}_H$ is a subfactor of $\mathcal{L}_{G_n}$. For any large integer m_0, choose projections $P_1, \ldots, P_{m_0}$ in the abelian von Neumann algebra generated by $L_{g_{31}}$ and $Q_1, \ldots, Q_{m_0}$ in the one generated by $L_{g_{32}}$ such that $\tau(P_j) = \tau(Q_j) = 1/m_0$ and $\Sigma_{j=1}^{m_0} P_j = \Sigma_{j=1}^{m_0} Q_j = I$. From *Lemma 3*, we know that $\tau(P_j Q_k) = 1/m_0^2$ and $\{P_j Q_k : j, k = 1, \ldots, m_0\}$ is a set of m_0^2 mutually orthogonal projections in $\mathcal{L}_H$. Let $m = m_0^2$ and choose a full matrix subalgebra $\mathcal{N}_m$ in $\mathcal{L}_H$ and a matrix unit system $\{E_{jj}\}_{j=1}^m$ so that the diagonal projections $E_{11}, \ldots, E_{mm}$ coincide with the projections $P_1Q_1, \ldots, P_1Q_{m_0}, \ldots, P_{m_0}Q_1, \ldots, P_{m_0}Q_{m_0}$. Because g_{21} commutes with g_{31} and g_{12} commutes with g_{32}, we have

$$L_{g_{21}} = \sum_{l=1}^{m_0} P_l L_{g_{21}} P_l = \sum_{l,k_1,k_2=1}^{m_0} P_l Q_{k_1} L_{g_{21}} Q_{k_2} P_l,$$

$$L_{g_{12}} = \sum_{k=1}^{m_0} Q_k L_{g_{12}} Q_k = \sum_{k,l_1,l_2=1}^{m_0} P_{l_1} Q_k L_{g_{12}} Q_k P_{l_2},$$

$$L_{g_{31}} = \sum_{l=1}^{m_0} P_l L_{g_{31}} P_l = \sum_{l,k=1}^{m_0} P_l Q_k L_{g_{31}} Q_k P_l,$$

$$L_{g_{32}} = \sum_{k=1}^{m_0} Q_k L_{g_{32}} Q_k = \sum_{k,l=1}^{m_0} P_l Q_k L_{g_{32}} Q_k P_l.$$

Thus with respect to the matrix unit system $\{E_{jk}\}_{j,k=1}^m$, the elements $L_{g_{21}}$, $L_{g_{31}}$, $L_{g_{32}}$ and $L_{g_{12}}$ have only $2m_0^3 + 2m_0^2$ nonzero entries. Choose m_1 so that $m_1(m_1 - 1)/2 \geq 2m_0^3 + 2m_0^2$ and $m_1 \leq 2m_0\sqrt{m_0 + 1} + 1$. By the same matrix trick used in the proof of *Lemma 4*, there is a selfadjoint element T in $\mathcal{L}_H$ whose support is majorized by $E_{11} + \cdots + E_{m_1m_1}$, such that $\mathcal{L}_H$ is generated by $\mathcal{N}_m$ and T. When m_0 is large, we have that the trace of the support of T is less than or equal to m_1/m, which can be arbitrarily small. Next, we may consider the subfactor of $\mathcal{L}_{G_n}$ generated by $\mathcal{L}_H$ and $L_{g_{23}}$. Continuing this process and by repeated use of *Lemmas 2* and *4*, we have that $\mathcal{L}_{G_n}$ is generated by a matrix subalgebra $M_{m_n}(\mathbf{C})$ and a selfadjoint element T_n with an arbitrarily small support P, which is dominated by some diagonal projections in $M_{m_n}(\mathbf{C})$. It is easy to see that $M_{m_n}(\mathbf{C})$ is generated by a rank one projection E_1 and a (shift) unitary matrix U_n (of finite order). One may choose E_1 so that it is orthogonal to P. Replacing E_1 and T_n by one selfadjoint element S, we have that $\mathcal{L}_{G_n}$ is generated by S and U_n, where S can have an arbitrarily small support and U_n is a unitary element with a finite order. This completes the proof of our theorem. ■

Note that when $n \geq 4$ even, $\mathcal{L}_{SL_n(\mathbf{Z})} \cong \mathcal{L}_{\mathrm{PSL_n}(\mathbf{Z})} \oplus \mathcal{L}_{\mathrm{PSL_n}(\mathbf{Z})}$. Thus the same result holds for $\mathcal{L}_{SL_n(\mathbf{Z})}$ for all $n \geq 3$.

Final Comments

As pointed out by Voiculescu (4), free entropy dimension should be related to the number of generators for a factor. The "number" should be measured by its free entropy dimension, not by the length of the support of the generator. To illustrate this point, we shall introduce a notion of "generating length" and show that a large class of factors have generating length equal to one.

Definition 6: Suppose $\mathcal{M}$ is a factor of type II_1 with trace τ or, in general, a von Neumann algebra with a state, and $\mathcal{N}$ a subalgebra (or subset) of $\mathcal{M}$. The generating length $\kappa_{\mathcal{N}}(\mathcal{M}, \tau)$ of $\mathcal{M}$ over $\mathcal{N}$ is given as follows:

$$\kappa_{\mathcal{N}}(\mathcal{M}, \tau) = \inf\{\tau(P_1) + \tau(P_2) + \cdots : \quad P_j \text{ is the range projection of } A_j, A_j = A_j^* \in \mathcal{M}, \mathcal{M} \text{ is generated by } \mathcal{N} \text{ and } A_1, A_2, \ldots\}.$$

When $\mathcal{N}$ is $\mathbf{C}I$ or empty set, we shall use $\kappa(\mathcal{M}, \tau)$ or simply $\kappa(\mathcal{M})$ instead of $\kappa_{\mathcal{N}}(\mathcal{M}, \tau)$. The advantage of this generating length is that it tells more information, in some sense, than the number of selfadjoint generators of a von Neumann algebra. The detailed study of this invariant will appear elsewhere. Using a matrix trick, one easily proves the following result.

PROPOSITION 7. *Suppose* $\mathcal{M}$ *is a factor of type* II_1 *and* P *is a projection of trace* $\frac{1}{2}$. *If* $P\mathcal{M}P$ *is generated by two selfadjoint elements, then* $\kappa(\mathcal{M}) = 1$.

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