Near strongly resonant periodic orbits in a Hamiltonian system

Vassili Gelfreich*

Mathematics Institute, University of Warwick, Coventry CV4 7AL, United Kingdom

Communicated by John N. Mather, Princeton University, Princeton, NJ, February 27, 2002 (received for review May 23, 2001)

We study an analytic Hamiltonian system near a strongly resonant periodic orbit. We introduce a modulus of local analytic classification. We provide asymptotic formulae for the exponentially small splitting of separatrices for bifurcating hyperbolic periodic orbits. These formulae confirm a conjecture formulated by V. I. Arnold in the early 1970s.

In Arnold’s Mathematical Methods of Classical Mechanics (1), it is explained that a linearly stable periodic orbit of a Hamiltonian system can be unstable due to the influence of nonlinear terms. In the 1970s, Brjuno (2) constructed corresponding resonant normal forms in the form of power series. The normal forms are integrable at all orders. The nonintegrability of the original system can be responsible for the divergence of the series. In the present note we consider Hamiltonian systems with two degrees of freedom. We introduce a symplectic invariant, a splitting constant, that measures the lack of integrability. This invariant characterizes the splitting of complex invariant manifolds.

In a Hamiltonian system periodic orbits are not usually isolated but form one-parametric families. Naturally the value of the Hamiltonian function \( H \) plays the role of the parameter. Thus even in the case when the original Hamiltonian does not explicitly contain any parameter, it is possible to observe bifurcations of periodic orbits. A bifurcation corresponds to a resonance between the frequency of a periodic orbit and the frequency of small oscillations around it. In a generic situation there is a family of hyperbolic periodic orbits of a multiple period, which shrinks to the resonant periodic orbit at an exact resonance. Separatrices of the hyperbolic periodic orbit have to intersect due to Hamiltonian nature of the problem. Segments of separatrices of the corresponding resonant normal form make up a closed loop around the periodic trajectory. In section E, appendix 7 of ref. 1, Arnold pointed out that there should be an important qualitative difference between the original Hamiltonian system and its normal form due to the splitting of separatrices. He also noticed “the magnitude of the splitting of separatrices is exponentially small. Nevertheless the error of the Melnikov method can be considered as a time-one map of an area-preserving map. Assume that there is a family of elliptic fixed points \( p_n \) with a (complex) multiplier \( \mu_n \). If \( |\mu_n| = 1 \) we say that \( p_n \) is resonant. In this article we study the strong resonances of orders \( n = 1, 2, 3 \), and 4. Similar to the non-Hamiltonian theory the case \( n = 4 \) has some special properties, and it is to be studied separately (5).

Bifurcations Near Strongly Resonant Periodic Orbits

The bifurcation picture near a resonant periodic orbit of an order \( n \) is well known. In this short section we follow the book (2). The \( n \)th iterate \( F_n^n \) can be considered as a time-one map of an autonomous Hamiltonian system with one degree of freedom with a very small error. It is possible to simplify the form of the corresponding Hamiltonian by a canonical transformation. In this way the system is transformed to a resonant normal form. It coincides with a normal form of an autonomous vector field with \( Z_n \) symmetry (6).

The lower order normal form Hamiltonian \( H_0 \) depends on two parameters: \( \delta \) describes deviation from the exact resonance (and

E-mail: gelf@maths.warwick.ac.uk.

www.pnas.org/cgi/doi/10.1073/pnas.212116699
it is proportional to a power of $e$, and $A$ depends on the map $F_0$ only. It is not too difficult to provide an explicit formula for $A$ and $\delta$ as a function of several first coefficients of the Taylor series of $F_\omega$. We say that the bifurcation is nondegenerate if $A$ and the lowest order coefficient of the series for $\delta(e)$ do not vanish. These formulae are not of big importance for the purposes of the present paper since we will formulate our results in terms of quantities, which allow coordinate independent definitions. The phase portraits and the main-order terms of the normal forms are given on Figs. 1 and 2. This is a complete list of nondegenerate cases for resonances of orders $n = 1$, $2$, and $3$ (2). The phase portrait of the map $F_\omega$ looks quite similar to its normal form, but still some explanations are necessary.

$n = 1$. The elliptic fixed point collides with a hyperbolic fixed point and disappears. This corresponds to a Hamiltonian saddle-center (Bogdanov-Takens) bifurcation (Fig. 1a; ref. 7). There is no (real) fixed point of the map $F_\omega$ after the collision.

$n = 2$. We see a pitch-fork period-doubling bifurcation. There are two possibilities:

- $A > 0$: The elliptic fixed point collides with a hyperbolic period-two orbit. After that it becomes hyperbolic, and the period-two orbit disappears (Fig. 1b).
- $A < 0$: The elliptic fixed point becomes hyperbolic, and a period-two elliptic orbit arises (Fig. 2).

$n = 3$. The elliptic fixed point exists on both sides of the resonance. At the resonance it collides with a period-three hyperbolic periodic orbit, which also exists on both sides from the resonance (Fig. 1c).

In each of these cases separatrices of the normal form make up a small loop around the elliptic equilibrium. This loop shrinks to a point at the exact resonance. At any order the resonant normal form is integrable, and the corresponding stable and unstable separatrices coincide. This is not generically true for the map $F_\omega$—as it was already pointed out in ref. 1—the separatrices of the map can intersect transversally. Detecting this splitting of separatrices is a very difficult analytical problem, because in the analytical case the splitting of separatrices is exponentially small in $e$. In particular, the Melnikov method does not generically give a correct answer in this case.

**Symplectic Invariant $\Theta$**

In this section we study an analytic area-preserving map $F_0$ near a strongly resonant periodic orbit, $p_0 = F_0(p_0)$. Without any loss of generality we may assume that $p_0$ is at the origin. We associate with $F_0$ a complex number $\Theta$. Its absolute value $|\Theta|$ is a modulus of local real-analytic symplectic classification. Moreover, $\Theta = \Theta(F_0)$ analytically depends on $F_0$, but it is not determined by any finite jet of $F_0$ at the origin.

We make the following assumptions:

[A1] $F_0$ is a real-analytic diffeomorphism of $(\mathbb{C}^2, 0)$, which preserves the complex area-form $du \wedge dv$. 

---

**Fig. 1.** Bifurcations of equilibria of the resonant normal forms ($A > 0$).

**Fig. 2.** Bifurcation of the equilibrium of the resonant normal form $[H_2 = \delta(x^2/2) + y^2/2 + A(x^4/4), A > 0]$. 

---
The Jacobian matrix $F_0(0)$ is generic.

The lower order coefficient $A$ of the normal form is positive.

The phase portraits of the corresponding normal forms are shown in the central column of Fig. 1, which corresponds to $\delta = 0$. The eigenvalues of the matrix $(F_0)'(0)$ satisfy $\mu_0 = 1$. If $n = 1$ or 2, they are doubled. The genericity assumption [A2] means that the matrix can be transformed to a Jordan block in the cases $n = 1, 2$, and it is diagonalizable in the case $n = 3$ (and also $n \approx 3$, but we do not consider these cases here).

The origin is a nonhyperbolic fixed point of the map $F_0^n$. It is possible to show (like refs. 8 and 9) that it has stable and unstable invariant manifolds $W^s$ and $W^u$—the set of all points which iterations converge to the origin,

$$W^\pm = \{ p \in \mathbb{C}^2 : F_0^n(p) \to 0, n \to \pm \infty \}. $$

We parameterize these manifolds by analytic solutions of the following finite-difference equation.

$$\begin{pmatrix} u \\ v \end{pmatrix}(\tau + 1) = F_0^n \left[ \begin{pmatrix} u \\ v \end{pmatrix}(\tau) \right]. $$

We supply this equation with the following asymptotic boundary conditions. The class of formal solutions in $\mathbb{C}[[\tau^{-1}]]$, the formal series in powers of $\tau^{-1}$ with real coefficients. The formal solution is not unique but defined upon a translation $\tau \to \tau + \tau_0$ with $\tau_0 \in \mathbb{R}$. The equation has two analytic solutions $\{u^+(\tau), v^+(\tau)\}$, which are asymptotic to the single formal solution as $\tau \to \pm \infty$, respectively. In this way one formal solution gives two different asymptotic behaviors, each of them uniquely defines the corresponding analytic solution. Now we consider analytical continuations in $\tau$ into two complex domains

$$D_+ = \{ \tau \in \mathbb{C} : \text{Arg}(\tau) > \alpha, |\tau| > R \}$$

with arbitrarily fixed $\alpha$ and sufficiently large $R$. The asymptotic property of $\{u^+(\tau), v^+(\tau)\}$ remains valid at infinity in $D_+$ respectively. The intersection

$$D_+ \cap D_- = \Pi_+ \cup \Pi_-$$

is a union of two connected components. Let us compare $u^+$ and $u^-$ at infinity in the lower component $\Pi_-$. They have the same asymptotic behavior as $\text{Im} \tau \to -\infty$. Thus $u^-(\tau) - u^+(\tau)$ decreases faster than any power of $\tau^{-1}$ as $\text{Im} \tau \to -\infty$. Moreover, it can be shown that it decreases exponentially. It is possible that this difference is identically zero.

The symplectic invariant, also called Lazutkin constant $\Theta$, is defined as the following limit.

$$\Theta = \lim_{\text{Im} \tau \to -\infty} e^{2\pi i \varepsilon} \det \begin{pmatrix} \frac{du}{d\tau} & u^+ - u^- \\ \frac{dv}{d\tau} & v^+ - v^- \end{pmatrix}(\tau). $$

Using the methods described in ref. 8 or 11, we can show that under the assumptions [A1]–[A3], the limit exists. The modulus of the splitting constant $|\Theta|$ is defined uniquely. In particular, it does not depend on the freedom in the choice of the normal form solution, because the choice of the constant $\tau_0 \in \mathbb{R}$ affects the argument of $\Theta$ only; $\Theta(F_0)$ is invariant with respect to analytic symplectic changes of coordinates, and it is an analytic function of $F_0$.

If the map $F_0$ has a nontrivial analytic integral, then $\Theta(F_0) = 0$. Because for any $N$ and any $F_0$ there is an integrable map that has the same $N$ jet as $F_0$, the splitting constant $\Theta(F_0)$ is not determined by any finite jet of $F_0$. On the other hand there are well developed numerical methods that can be used for computing $\Theta(F_0)$ for a given $F_0$ (see, e.g., refs. 9–11). High-precision computations are also described in ref. 12.

We conjecture that for any map $F_0$, any $\Theta \in \mathbb{C}$ and any natural $N$ there is another map $F_0$, which has the same Taylor expansion as $F_0$ up to order $N$, and $\Theta(F_0) = \Theta_0$.

There is an analytic proof that in the case of the area-preserving Hénon map and $n = 1$ the splitting constant doesn’t vanish (11). Consequently $\Theta$ does not vanish generically due to analyticity. Thus transformations to the resonant normal forms are generically divergent.

Finally, usually $\Theta(F_0)$ cannot be computed by the Melnikov method.

### Splitting of the Small Separatrix Loop

In this section we provide asymptotic formulae for the splitting of the small separatrix loop near a generic strongly resonant periodic orbit. The separatrix splitting is exponentially small compared with a natural bifurcation parameter. These formulae cannot be obtained by the Poincaré–Melnikov method.

We need one more definition first. There are many different ways in which one may quantitatively describe the separatrices splitting. We prefer Lazutkin homoclinic invariant, which is defined in the following way. Let the separatrices of the hyperbolic $n$-periodic orbit be parameterized by solutions of the following finite-difference equation

$$\psi(t + h) = F_0^n[\psi(t)], \quad \psi(t) \to x_0 \quad \text{as} \ t \to \pm \infty,$$

where $x_0$ is one of the $n$ points of the hyperbolic periodic trajectory. The “$+$” corresponds to the stable separatrix, and the “$-$” corresponds to the unstable one. We assume that the multiplier of the hyperbolic periodic orbit $\lambda_n > 1$, and let $h = \log \lambda_n$. Under an additional condition (2 $\pi n$ periodicity in $t$) these solutions are unique up to a substitution $t \to t + \tau_0$. The image $\psi^* = \psi(t)$ of $\psi(t)$ in $\mathbb{C}^2$ is a vector field tangent to the stable and unstable separatrices, respectively. These vector fields are $\mathcal{C}^0$-invariant. It is easy to see that the vector fields are independent from the choice of the constant $\tau_0$. Let $x_0$ denote a homoclinic point. Then Lazutkin homoclinic invariant, $\omega := \varepsilon \cdot \bar{\omega} + \varepsilon$, equals the area of a parallelogram defined by the tangent vectors $\bar{\omega}^-$ and $\bar{\omega}^+$ at $x_0$. The Lazutkin homoclinic invariant has remarkable properties: (i) it takes the same value for all points of the homoclinic trajectory of $x_0$, (ii) it is invariant with respect to canonical substitutions, and (iii) for a fixed $x_0$ it is proportional to the sine of the angle between separatrices—the splitting angle.

We have chosen the homoclinic invariant to describe the splitting by one invariantly defined number, which is easy to compute with high precision. This permits a careful comparison of the theory with results of numerical experiments. Of course, inside the proofs there are expressions that describe the behavior of the separatrices as curves, i.e., much detailed description of the separatrices is also possible by the same method.

We say that a homoclinic point is primary if it is the “first” intersection of separatrices. Its trajectory is called a primary homoclinic orbit. Sometimes a primary homoclinic orbit is also called a “one-bump homoclinic.” The main result of the present paper is the following asymptotic formula for the Lazutkin homoclinic invariant of a primary homoclinic orbit.

**Theorem 1.** Let the map $F_0$ satisfy [A1]–[A3] from the previous section and the map $F_1$ be analytic in all its variables in a neighborhood of the origin. Let the map $F_0$ afford one of the nondegenerate resonant normal forms shown on Fig. 1. Assume $\varepsilon$ be sufficiently small, then the following is true. The map $F_0$ has exactly one $n$-periodic hyperbolic trajectory in a neighborhood of $p_0$. 

PNAS | October 29, 2002 | vol. 99 | no. 22 | 13977

Gelfreich
where $h = \log \lambda_e$ and $\lambda_e$ is the multiplier of the hyperbolic periodic orbit.

The current note does not contain the proof of this theorem. The case of $n = 1$ was considered in ref. 8. A complete and detailed exposition of the method applied to the standard map is contained in ref. 8.

It is not too difficult to check using the resonant normal forms described in Bifurcations Near Strongly Resonant Periodic Orbits that $h$ is of order of some power of $\varepsilon$ (depending on $n$). Thus $h$ vanishes together with $\varepsilon$, and the splitting of separatrices predicted by Eq. 3 is exponentially small compared with $\varepsilon$, too. The preexponential factor $\Theta = \Theta(F_0)$ is the splitting constant, defined in the previous section. It depends on the map at the moment of the exact resonance only. If $\Theta$ does not vanish, there are exactly two primary homoclinic orbits with different orientation of intersection of separatrices, and both of them are transversal. The splitting constant $\Theta$ is an important ingredient of the asymptotic formulae. In particular, the asymptotic formulae imply transversality of the separatrices only provided $\Theta$ does not vanish.

The constant $\Theta$ can vanish if, for example, the map $F_0$ is exactly time one map of the corresponding normal form or if $F_0$ is integrable. Numerical experiments show that $\Theta$ may vanish in nonintegrable cases, too. If $\Theta$ vanishes, Theorem 1 gives only an upper bound for the splitting of separatrices, and we can’t exclude the possibility that the separatrices do not split at all.

Theorem 1 does not cover the case of a resonance of the order $n = 2$ with the resonant normal form shown on Fig. 2.

**Theorem 1′.** Let the map $F_0$ satisfy the assumptions of Theorem 1 except that its second-order resonance has the resonant normal form shown on Fig. 2. After the bifurcation, the map $F_0$ has exactly one two-periodic elliptic orbit in a neighborhood of $p_e$. Separatrices of $p_e$ form small loops around the elliptic periodic orbit. There are at least two different primary homoclinic orbits. The homoclinic invariant of one primary homoclinic orbit is given asymptotically by

$$\omega = \frac{4\pi}{h^2} e^{-\frac{2\pi^2}{h^2} \left[ |\Theta| + O(h) \right]}$$

where $h = \log \lambda_e$, $\lambda_e < -1$ is the multiplier of $p_e$, which becomes a hyperbolic fixed point after the bifurcation.

The only essential difference compared with Theorem 1 is in the exponent in Eq. 4.

If the map $F_0$ has additional symmetries, its separatrix splitting can be much smaller compared with the above-described general case. For example, consider $n = 2$ and let the map $F_0$ be odd. Then the constant $\Theta(F_0) = 0$ due to the symmetry, and the splitting of separatrices is essentially smaller compared with the general non-odd case. Nevertheless an asymptotic formula for the splitting of separatrices can be derived by the same method for this case, too. Instead of $F_0$ we have to consider an auxiliary map $G_0 = -F_0$. Note that $G_0^2 = F_0^2$, so the separatrices of these two maps coincide. The corresponding statements and asymptotic formula (Eq. 4) remain valid for $G_0$, if we use $h = \log(-\lambda_e)$. Note that the multiplier of the hyperbolic fixed point of $G_0$ equals $-\lambda_e$. Thus the new $h$ is a half of the corresponding value for $F_0$, and the order of the splitting is much smaller (approximately square of the general case); moreover, if $\Theta(F_0) \neq 0$, then the map $F_0$ has exactly four different primary homoclinic orbits.

**Final Remarks**

In many cases an exponential upper bound for the size of the chaotic zone can be derived from Neishtadt’s averaging theorem (13). A sharper upper bound for the splitting of separatrices can be derived easily from ref. 14. The exponentially small confinement of chaos is not purely a Hamiltonian phenomenon (15).

To establish the transversality of separatrices it is necessary to know lower bounds or asymptotic formulae for the splitting of separatrices. This problem is delicate and complicated. An example of a degenerate Hamiltonian bifurcation was considered in the pioneer paper in ref. 4: It was shown that the splitting of separatrices can be detected by the Melnikov method (see also ref. 16). In particular, in this degenerate case the constant $\Theta$ can be computed analytically by the Melnikov method.

In the general case, described in the present note, as far as the author knows, all attempts of using modifications of the Melnikov method have failed, although recent numerical studies (17) showed that there are a few comparatively simple degenerate examples where the Melnikov method provides an adequate estimate for the splitting.

**Theorems 1 and 1′ and the statements of Symplectic Invariant $\Theta$ are proved by Lazutkin’s method, proposed by Lazutkin (18) for studying the exponentially small splitting of separatrices for the standard map in 1984. Lazutkin’s original proof was incomplete, and the method required substantial development by Lazutkin himself and his collaborators (3) until a complete proof of Lazutkin’s original asymptotic formula was finally published in ref. 8. The corresponding proof is rather long. An informal derivation of the asymptotic formula for the case of a first-order resonance can be found in ref. 9.

According to Lazutkin’s method, the proofs of Theorems 1 and 1′ are based on an accurate study of the analytic continuation of the map $F_0$ and its separatrices into the complex phase space. In a complex domain the splitting of separatrices is not small and can be detected by a specially developed complex matching method. The splitting of complex separatrices is already present for $F_0$ at the exact resonance, although it is not visible on the real pictures but in a complex domain only. The value of this separatrix splitting is described by the complex constant $\Theta$, and it is inherited near the resonance. In this sense, $|\Theta|$ plays the role of “separatrix value” usual in the bifurcation theory. A “discrete flow box theorem” (‘‘theorem on analytic integral’’) is used to come back from complex dynamics to the real one.

As always the strong resonance of order $n = 4$ is more complicated, and its study is not covered by the present note. The splitting constant $\Theta$ can be defined quite similarly. I conjecture that in this case an asymptotic formula for the Lazutkin homoclinic invariant is different from Eq. 3, and some bifurcations of homoclinic orbits are to be expected.

This paper is dedicated to the memory of my teacher Vladimir Lazutkin. An essential part of the present work was done at the Mathematical Institute of the Free University Berlin. I thank B. Fiedler and L. Lerman for stimulating discussions. This work has been partially supported by the INTAS Grant 00-221.