Non-Lipschitz minimizers of smooth uniformly convex functionals

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We construct non-Lipschitz minimizers of smooth, uniformly convex functionals of type $I(u) = \int_\Omega f(Du(x))dx$. Our method is based on the use of null Lagrangians.

1. Introduction

We consider variational integrals of the form

$$I(u) = \int_\Omega f(Du(x))dx,$$

where $\Omega$ is a bounded open set with smooth boundary in $\mathbb{R}^n$, $u: \Omega \to \mathbb{R}^m$ $Du$ is the gradient matrix of $u$, and $f: M_{m \times n}^{m \times n} \to \mathbb{R}$ is a smooth uniformly convex function with uniformly bounded second derivatives. Here $M_{m \times n}^{m \times n}$ denotes the set of real $m \times n$ matrices. [Recall that we say $f$ is uniformly convex if there exists a constant $\rho > 0$ such that for all $\xi \in M_{m \times n}^{m \times n}$, $X \in M_{m \times n}^{m \times n}$, the inequality $f_{ij,\rho}(X)\xi \xi_n \geq \rho|\xi|^2$ holds.]

We shall consider the regularity of minimizers of $I$ belonging to $W^{1,2}(\Omega, \mathbb{R}^m)$. By a minimizer we mean a mapping $u \in W^{1,2}(\Omega, \mathbb{R}^m)$ such that for any smooth mapping $\phi: \Omega \to \mathbb{R}^m$ compactly supported in $\Omega$ the inequality $I(u + \phi) \geq I(u)$ holds. When $f$ is uniformly convex with uniformly bounded second derivatives, it is not difficult to see that $u$ is a minimizer of $I$ if and only if $u$ is a weak solution of the Euler–Lagrange equation of $I$, i.e., $u$ is a weak solution of

$$\partial_u f_i(Du(x)) = 0, \quad i = 1, \ldots, m. \tag{1.2}$$

(Here and in what follows we use the summation convention.)

A classical result of Morrey (see ref. 1) shows that when $n = 2, m \geq 1$, and $f$ is a smooth uniformly convex function with uniformly bounded second derivatives, every weak solution of Eq. 1.2 is smooth; this is also the case when $n = 2, m = 1$, and $f$ satisfies the same condition by fundamental work of De Giorgi (see ref. 2) and Nash (see ref. 3). The method used in the proof of De Giorgi and Nash cannot be extended to the case $m \geq 2$ as shown by a counterexample of De Giorgi (see ref. 4). The first example of a nonsmooth minimizer for a smooth uniformly convex functional of type 1.1 was constructed by Nečas in high dimensions (see ref. 5). He considered $u: \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$u_{ij} = \frac{x_i x_j}{|x|}, \tag{1.3}$$

and for large $n$ constructed a smooth uniformly convex function $f$ with bounded second derivatives defined on $M_{2 \times 2}^n$, for which $u$ is a minimizer of the corresponding functional $I$. Later, Hao et al. (6) were able to modify this construction and make it work for $n \geq 5$. They modified the original $u$ in the following way:

$$u_{ij} = \frac{x_i x_j}{|x|} - \frac{|x|}{n} \delta_{ij}. \tag{1.4}$$

Recently we (see ref. 7) constructed a nonsmooth minimizer of a smooth uniformly convex functional of type 1.1 in the case $n = 3, m = 5$ by considering the same function $u$ defined by Eq. 1.4. The main idea of our construction is the following. Let $K = \{\nabla u(x), x \in \Omega\}$ be the set of gradients of $u$. We find a null Lagrangian $L$ (see Definition 2.1) such that

$$\nabla L(X) = \nabla f(X), \quad \forall X \in K, \tag{1.5}$$

for a smooth uniformly convex function $f$ with bounded second derivatives. Then $u$ will satisfy the Euler–Lagrange equation of $I$ automatically.
All the counterexamples of nonsmooth minimizers above are Lipschitz-continuous. In fact, it was an open problem whether minimizers with unbounded gradients exist. Partial results in this direction can be found in ref. 8, where local Lipschitz continuity of minimizers for a special class of functionals was obtained.

In this paper we use the null Lagrangian approach to construct counterexamples showing, among other things, that in general for \( n \geq 3 \) we cannot expect Lipschitz continuity of the minimizer of a smooth uniformly convex functional. Moreover, for \( n = 5 \) we find a locally unbounded solution to Eq. 1.2. We recall that \( n = 5 \) is the first possible dimension where such an example is possible. (When \( n \leq 4 \) each minimizer must be Hölder-continuous, because it belongs to \( W^{2,2+\delta} \) for some \( \delta > 0 \); see ref. 9.) We also construct in section 4 a completely new example for \( n = 4, m = 3 \). The important feature in this example is the low dimension of the target space. The construction also gives a non-Lipschitz minimizer in this case. The mapping used in that example is derived from the Hopf fibration \( S^3 \to S^2 \), which can be thought of as a complex version of Eq. 1.4. In addition, as a byproduct of our methods, we find an example (with \( n = m = 3 \)) of nonuniqueness of weak solutions of Eq. 1.2 in the spaces \( W^{1,p} \) with \( 1 < p < 2 \). This is briefly explained in section 5.

For counterexamples to regularity of solutions of elliptic systems that are not of the form of Eq. 1.2 we refer the reader to refs. 23 and 25. Examples of non-smooth unbounded minimizers of functionals of the form of Eq. 1.1 for integrands with unbounded second derivatives (the so-called \( p, q \)-growth conditions) were obtained even in the scalar case in refs. 10–12. A comprehensive treatment of regularity questions can be found in ref. 9. Interesting sufficient conditions for regularity are discussed in ref. 13.

2. Preliminaries

First we introduce some basic facts about null Lagrangians.

Definition 2.1 (see ref. 14): \( L : M^{m \times n} \to \mathbb{R} \) is a null Lagrangian if for each smooth \( u : \mathbb{R}^n \to \mathbb{R}^m \),

\[
\text{div} L(u(x)) = 0.
\]

[2.1]

We recall the following classical theorem about null Lagrangians (see refs. 15 or 16).

Proposition 1. Let \( L : M^{m \times n} \to \mathbb{R} \), the following conditions are equivalent:

(i) \( L \) is a null Lagrangian.

(ii) \( L \) is a linear combination of subdeterminants.

(iii) \( L \) is rank-one affine, i.e., \( t \to L(A + tB) \) is affine for each \( A \in M^{m \times n} \) and each \( B \in M^{m \times n} \) with rank \( B = 1 \).

Moreover, if \( L \) is quadratic, then any of the above conditions are satisfied if and only if \( L(B) = 0 \) for each \( B \in M^{m \times n} \) satisfying rank \( B = 1 \).

3. The Case \( n \geq 3, m = n(n + 1)/2 - 1 \)

Let \( \Omega \) be the unit ball in \( \mathbb{R}^n \). Consider \( u^*(x) = (u^*_{ij}(x)) \) given by

\[
u_{ij}(x) = \frac{x_i x_j}{|x|^2} - \frac{|x|^2}{n} \delta_{ij}, \quad u^*_{ij}(x) = |x|^2 u_{ij}(x) = x_i x_j, \quad i, j = 1, \ldots, n.
\]

[3.1]

Then for each \( x \in \Omega, u^*(x) \in \{ A \in M^{m \times n}, A = 0, \text{Tr } A = 0 \} \equiv \mathbb{R}^{n(n+1)/2-1}. \) For each \( R \in SO(n) \) we have

\[
u^*(Rx) = Ru^*(x)R^t.
\]

Denote \( K^0 = \{ \nabla u^*(x), x \in \Omega \}, K^1 = \{ \nabla u^*(x), x \in S^{n-1} \}. \) Following ref. 7, we identify \( M^{m \times n} \) with \( T = \{ a_{ijk} \in (\mathbb{R}^n)^{n^2}; a_{ijk} = a_{jik} = a_{ikj} = 0 \} \) in the obvious way. We recall that \( m = [n(n + 1)/2] - 1 \) and we use a classical procedure to decompose \( T \) into irreducible subspaces (see ref. 17). We first decompose \( T \) into the trace-free part \( T' \) and its orthogonal supplement \( T_0 \), i.e., \( T = T' \oplus T_0 \). An easy calculation shows that the projection on \( T_0 \) is given by \( a_{ijk} \to -[2/(n+2)(n-1)]\delta_{ij}\eta_k + [n/(n+2)(n-1)]\delta_{ik}\eta_j \) with \( \eta_k = a_{kij}, k = 1, \ldots, n \). Then we decompose \( T' \) by using symmetrizations. We have \( T' = T_1 \oplus T_2 \), where the projection on \( T_1 \) is given by symmetrizations, i.e., \( a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{ijk} + a_{ikj}) \); the projection on \( T_2 \) is given by \( a_{ijk} \to \frac{1}{3}(a_{ijk} + a_{ikj} - a_{jk} - a_{kj}) \), which corresponds to the following Young tableau.

\[
\begin{array}{|c|c|c|}
\hline
1 & 2 & 3 \\
\hline
\end{array}
\]

We remark that the antisymmetric part of any tensor in \( T \) is 0.

By the above formula, a rank one matrix \( a_{ijk} = C_{ijk}^k \) in \( M^{m \times n} \) with \( C = C^k \), \( \text{Tr } C = 0 \) can be decomposed as

\[
a_{ijk} = a_{ijk}^1 + a_{ijk}^2 + a_{ijk}^3,
\]

with

\[
|a_{ijk}^1|^2 = \frac{1}{2}|C^i|^2 |g|^2 + \frac{2n}{3(n+2)} |C^i|^2, \quad |a_{ijk}^2|^2 = \frac{2}{3} |C^i|^2 |g|^2 - \frac{2n}{3(n-1)} |C^i|^2, \quad |a_{ijk}^3|^2 = \frac{2n}{(n+2)(n-1)} |C^i|^2.
\]
For $X = X_1 + X_2 + X_3, X_i \in T$, we let $L(X) = -2|X_1|^2 + |X_2|^2 + n|X_3|^2$. From the above formula we see that $L$ vanishes on all rank-one matrices in $M^{m \times n}$, hence $L$ is a quadratic null Lagrangian on $M^{m \times n}$. Moreover, we have the following lemma.

**Lemma 3.1.** We have $L(X) = l_v = [2(n - 1)/(n + 2)|x_1 - 1 - x |^2 - (1 + x)^2]$ on $K^*_1$ and for

$$
0 \leq x < \frac{3(n + 1)}{n - 1} + 1,
$$

there exists constant $\delta_0(x) > 0$, such that for any $X = \nabla u^*(x), Y = \nabla u^*(y) \in K^*_1$, we have

$$
\nabla L(X)(Y - X) \equiv -\delta_0(x)|Y - X|^2.
$$

**Proof:** First we note that on $K^*_1$ we can decompose $\nabla u^*(x) = \{u^*_{ij,k}\}$ as follows:

$$
u^*_{ij,k}(x) = u^*_{ij,k} + u_{ij,k}^2 + u_{ij,k}^3, \quad x \in S^{n-1},
$$

where $u^* \in T_1$ with

$$
u^*_{ij,k} = 1 + \varepsilon \left[ x_i x_j x_k + \frac{1}{n + 2} (x_i \delta_k + x_j \delta_k + x_k \delta_i) \right],
$$

$$u_{ij,k}^2 = 0,
$$

$$u_{ij,k}^3 = \frac{n + 1 - \varepsilon}{n + 2} \left[ x_i \delta_k + x_j \delta_k - \frac{2}{n} \delta_j x_k \right]
$$

and

$$|u_{ij,k}^2| = \frac{(1 + \varepsilon)^2(n - 1)}{n + 2}, \quad |u_{ij,k}^3| = \frac{2(n + 1 - \varepsilon)^2(n - 1)}{n(n + 2)}.
$$

Hence $\forall x \in S^{n-1},$

$$L(\nabla u^*(x)) = \frac{2(n - 1)}{n + 2} \left[ (n + 1 - \varepsilon)^2 - (1 + \varepsilon)^2 \right] = l_v,
$$

$$|\nabla u^*(x)|^2 = \frac{n - 1}{n + 2} \left[ (1 + \varepsilon)^2 \right] = N_v,
$$

$$|\nabla L(\nabla u^*(x))|^2 = 8 \frac{n - 1}{n + 2} \left[ 2(1 + \varepsilon)^2 + n(n + 1 - \varepsilon) \right] = m_v^2.
$$

Because $L$ is quadratic, we have

$$L(\nabla u^*(x) - \nabla u^*(y)) = 2L(\nabla u^*(x)) - 2L(\nabla u^*(x), \nabla u^*(y)),
$$

where we also use $L$ for the symmetric bilinear form corresponding to the quadratic form $L$.

$$L(\nabla u^*(x), \nabla u^*(y)) = -2u_{ij,k}^1(x) u_{ij,k}^1(y) + m u_{ij,k}^3(x) u_{ij,k}^3(y)
$$

$$= -2(1 + \varepsilon)^2 \left[ x_i x_j x_k + \frac{1}{n + 2} (x_i \delta_k + x_j \delta_k + x_k \delta_i) \right] \left[ y_i y_j y_k + \frac{1}{n + 2} (y_i \delta_k + y_j \delta_k + y_k \delta_i) \right]
$$

$$+ n \left( \frac{n + 1 - \varepsilon}{n + 2} \right)^2 \left[ x_i \delta_k + x_j \delta_k - \frac{2}{n} x_k \delta_j \right] \left[ y_i \delta_k + y_j \delta_k - \frac{2}{n} y_k \delta_j \right]
$$

$$= -2(1 + \varepsilon)^2(x, y)^2 + \left( \frac{6}{n + 2} (1 + \varepsilon)^2 + \frac{2(n + 1 - \varepsilon)^2(n - 1)}{n + 2} \right)(x, y).
$$

Let $t = (x, y)$. Then $-1 \leq t \leq 1$, and we have

$$L(\nabla u^*(x) - \nabla u^*(y)) = 2L(\nabla u^*(x)) - 2L(\nabla u^*(x), \nabla u^*(y)).$$

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Lemma 3.2. When \( \mu \) is sufficiently small we have

\[
\nabla L(X) \cdot (\tilde{Y} - X) \leq -\frac{\delta_0(\varepsilon)}{2} |\tilde{Y} - X|^2, \tag{3.4}
\]

for each \( X \in K^*_1 \) and each \( \tilde{Y} \in B_{1/2 \mu} \), where \( B_{1/2 \mu} \) is defined above, with \( Y \) being an arbitrary point of \( K^*_1 \).

Proof: The inequality

\[
|\tilde{Y} - Y|^2 \leq \mu^2 |\nabla L(Y)|^2
\]

gives

\[
\nabla L(Y) \cdot (\tilde{Y} - Y) \leq -\frac{1}{2 \mu} |\tilde{Y} - Y|^2.
\]

Hence,

\[
\nabla L(X) \cdot (\tilde{Y} - X) = (\nabla L(X) - \nabla L(Y)) \cdot (\tilde{Y} - Y) + \nabla L(Y) \cdot (\tilde{Y} - Y) + \nabla L(X) \cdot (Y - X)
\]

\[
\leq (4 + 2n) |Y - X| |\tilde{Y} - Y| - \frac{1}{2 \mu} |\tilde{Y} - Y|^2 - \delta_0(\varepsilon) |Y - X|^2,
\]

and the statement follows easily.

Let \( S^e = \bigcup_{X \in K^*_1} B_{1/2 \mu}(X) \). When \( \mu \) is small, the boundary of \( S^e \) is smooth by elementary results about tubular neighborhoods (see ref. 18 or 19). Lemma 3.2 implies that (for sufficiently small \( \mu \)) all the eigenvalues of the second fundamental form of \( \partial S^e \) is negative and bounded above uniformly on \( K^*_1 \) by a negative constant \( \gamma_e \) [i.e. the principle curvatures \( k(X) \leq \gamma_e < 0 \), \( \forall X \in K^*_1 \)]. Because \( \partial S^e \) is smooth, we conclude that \( \partial S^e \) is locally strongly convex at any point of \( U^e \cap \partial S^e \), where \( U^e \) is a small neighborhood of \( K^*_1 \) in \( T_1 \oplus T_3 \).

Now take \( G^e \) to be the convex hull of \( S^e \) in \( T_1 \oplus T_3 \). Using Lemma 3.2 and the fact that \( \partial S^e \) is smooth and locally strongly convex in \( U^e \cap \partial S^e \), we infer that \( U^e \cap \partial G^e = U^e \cap \partial S^e \) when the neighborhood \( U^e \) of \( K^*_1 \) is chosen to be sufficiently small.
Let

\[ F^*_1(X) = \min \{ t \geq 0, X \in tG^* \}, \quad F^*(X) = I_t(F^*_1(X))^2. \]

Then \( F^* \) is two-homogeneous, smooth in \( U^e \), and uniformly convex in \( U^e \) (see ref. 20), and \( \nabla L(X) = \nabla F^*(X) \) for each \( X \in K^*_1 \).

Let \( \phi \) be a nonnegative \( C^\infty \) function with support in \([\frac{1}{2}, 1] \) such that

\[ \int_{\mathbb{R}^m} \phi(|Z|)dZ = 1. \]

Define

\[ \tilde{F}^*_\delta(X) = \left( \int_{\mathbb{R}^m} F^*_1(X + |X|Z\phi_\delta(|Z|)dZ \right)^2, \]

where \( \phi_\delta(X) = \delta^{-n}\phi(X/\delta) \). Then \( \tilde{F}^*_\delta \) is convex and two-homogeneous (see ref. 21). Let

\[ H^*_\delta,\tau(X) = \tilde{F}^*_\delta + \tau|X|^2, \quad X \in T_1 \oplus T_3. \]

Let \( \eta^*(X) \) be a smooth cutoff function defined on \( T_1 \oplus T_3 \), which vanishes outside \( U^e \) and is 1 in a smaller neighborhood \( U^* \) of \( K^*_1 \).

Consider

\[ \eta^*(X) = \tilde{\eta}^*(N_\ast X/|X|) \quad \text{when } |X| \neq 0. \]

Define

\[ G^*_{\delta,\tau}(X) = \begin{cases} (1 - \eta^*(X))H^*_\delta,\tau(X) + \eta^*(X)F^*(X) & |X| \neq 0, \\ 0 & X = 0. \end{cases} \]

A direct calculation shows that when \( \delta, \tau \) are small enough, \( G^*_{\delta,\tau} \) is two-homogeneous, smooth away from 0, and uniformly convex away from 0 with

\[ \nabla G^*_{\delta,\tau}(X) = \nabla L(X), \quad X \in K^*_1. \]  \[ \text{[3.5]} \]

Moreover, by the fact that

\[ G^*_{\delta,\tau}(\lambda X) = \lambda^2 G^*_{\delta,\tau}(X) \quad \forall \lambda \geq 0, \]

we see Eq. 3.5 holds on \( K^* \).

Fix \( \delta, \tau \) such that \( h^*(X) = G^*_\delta,\tau(X) \) is smooth and strongly convex away from 0. Let \( \psi(x) \) be a smooth mollifier defined on \( \mathbb{R}^m \) with support in \( B_1 \), and let \( \lambda^*(X) \) be a smooth cutoff function defined on \( \mathbb{R}^m \) such that \( \lambda^*(X) = 1 \) in \( B_{N_/4} \) and vanishes outside \( B_{N_/2} \). Define

\[ f^*_{\alpha,\beta}(X) = \lambda^*(X)(h^*_\alpha(X) + \beta|X|^2) + (1 - \lambda^*(X))h^*(X), \]

where \( h^*_\alpha(X) = h^* * \psi_\alpha(X), \psi_\alpha(X) = \alpha^{-n}\psi(X/\alpha) \). It is not difficult to see that when \( \alpha, \beta \) are small enough, \( f^*_{\alpha,\beta} \) is smooth and uniformly convex and satisfies Eq. 3.5 on \( K^* \).

In the last step, define

\[ f^*(A) = f^*_{\alpha,\beta}(X + Y) + |Z|^2, \]

where \( A = X + Y + Z, X \in T_1, Z \in T_2, Y \in T_3 \). Then \( f^* \) is a smooth, strongly convex function with bounded second derivatives, which satisfies Eq. 1.5 on \( K^* \).

We remark that we can take \( \epsilon > 1 \) when \( n \geq 5 \) and thus get an unbounded minimizer of a functional of the type 1.1.

**4. The Case \( n = 4, m = 3 \)**

In this section, let \( \Omega \) be the unit ball in \( \mathbb{R}^4 \). For \( \epsilon > 0 \), consider mapping \( \nu^\epsilon : \mathbb{R}^4 \to \mathbb{R}^3 \) given by

\[ \nu^\epsilon(z, w) = \left( \Re(zw), \Im(zw), \frac{|w|^2 - |z|^2}{2\epsilon^2 + 1} \right), \]

where \( (z, w) \in \mathbb{C}^2 \equiv \mathbb{R}^4, r^2 = |z|^2 + |w|^2, \) and \( \Re, \Im \) denote the real and imaginary part of \( f \), respectively.
For $R = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in SU(2)$, $|a|^2 + |b|^2 = 1$, $a, b \in \mathbb{C}$, denote by $\rho(R)$ the real four-dimensional representation of $SU(2)$ given by

$$
\rho(R)x = \begin{pmatrix}
 a_0 & -a_1 & b_0 & -b_1 \\
 a_1 & a_0 & b_1 & b_0 \\
 -b_0 & -b_1 & a_0 & a_1 \\
 b_1 & -b_0 & -a_1 & a_0
\end{pmatrix}
x_i, \quad x \in \mathbb{R}^4, \quad z = x_1 + ix_2, \quad w = x_3 + ix_4.
$$

Where

$$a = a_0 + ia_1, \quad b = b_0 + ib_1.$$

For $R \in SU(2)$, we have

$$v^r(\rho(R)x) = \tilde{\rho}_3(R)v^r(x), \quad x \in \mathbb{R}^4,$$

where $\tilde{\rho}_3(R)$ is the real representation of $SU(2)$ on $\wedge^3 \mathbb{R}^2$ induced by $\rho_3(R)$, the three-dimensional irreducible representation of $SU(2)$. We remark that $\rho_3$ is of the real type. We have

$$\nabla v^r(\rho(R)x) = \tilde{\rho}_3(R)\nabla v^r(x)(\rho(R)), \quad x \in \mathbb{R}^4,$$

where $\nabla f$ denotes $(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3, \partial f/\partial x_4)$.

In another way, we can write Eq. 4.2 as

$$\nabla v^r(Rx) = \tilde{\rho}_3(R) \otimes \rho(R)\nabla v^r(x)$$

for $x \in \mathbb{R}^4$. We then have the following lemma.

**Lemma 4.1.** There exists a unique (up to a multiplication by a real scalar) quadratic null Lagrangian on $M^{4 \times 3}$, which is invariant on the above action of $SU(2)$.

**Proof:** We identify the quadratic null Lagrangians on $M^{4 \times 3}$ with $\wedge^2 \mathbb{R}^4 \otimes \wedge^3 \mathbb{R}^3 \equiv \text{Hom}(\wedge^2 \mathbb{R}^4, \wedge^3 \mathbb{R}^3)$ and consider the representation $\tau$ of $SU(2)$ on $\text{Hom}(\wedge^2 \mathbb{R}^4, \wedge^3 \mathbb{R}^3)$ induced by $\rho_3 \otimes \rho$. Using elementary-group representation theory (see ref. 22 or 24), we can easily determine that over the field of complex numbers, $\tau$ decomposes into irreducible representations as

$$\tau = \rho_3 \otimes 4 \rho_3 \otimes \rho_1,$$

where $\rho_i$ denotes the unique $i$-dimensional irreducible representation of $SU(2)$ over $\mathbb{C}$. Because all $\rho_i$ appearing in this decomposition are of the real type, we can have the same decomposition over the real numbers. The statement follows easily.

A straightforward calculation along standard lines gives the following expressions for the null Lagrangian from Lemma 4.1:

$$L(X) = -X_{1212} + X_{1234} + X_{1313} + X_{1324} - X_{2323} + X_{2314}.$$  [4.4]

We recall that the invariant null Lagrangian is defined only up to a multiplication by a real scalar. In what follows we will use the normalization given by Eq. 4.4.

Now we follow the same method used in section 3 to construct the convex function $f^e$. First we have the following lemma.

**Lemma 4.2.** For $0 \leq e \leq \sqrt{7} - 2$, $x, y \in S^3$, $L(\nabla v^r(x)) = 2 - e$, and there exists constant $c_0(e) > 0$ such that

$$\nabla L(\nabla v^r(x)) - \nabla v^r(x) \leq -c_0(e)|\nabla v^r(x) - \nabla v^r(y)|^2.$$  [4.5]

**Proof:** When $r = 1$,

$$X_{1212} = -(x_3^2 + x_4^2) + (1 + e)(x_1^2 + x_2^2)(x_3^2 + x_4^2),$$

$$X_{1234} = x_1^2 + x_2^2 - (1 + e)(x_3^2 + x_4^2)(x_3^2 + x_4^2),$$

$$X_{1313} = x_3^2 + x_4^2 - x_1^2 - x_2^2 \frac{(x_3^2 + x_4^2 - x_1^2 - x_2^2)(1 + e)}{2} - 2(1 + e)x_1x_2(x_1x_3 + x_2x_4),$$

$$X_{1324} = x_3^2 + x_4^2 - (x_1^2 - x_2^2) \frac{(x_3^2 + x_4^2 - x_1^2 - x_2^2)(1 + e)}{2} - 2(1 + e)x_2x_4(x_1x_2 + x_3x_4),$$

$$X_{2323} = -x_1^2 - x_2^2 + (x_3^2 - x_4^2) \frac{(x_3^2 + x_4^2 - x_1^2 - x_2^2)(1 + e)}{2} - 2(1 + e)x_2x_4(x_1x_2 + x_3x_4),$$

$$X_{2314} = x_1^2 + x_2^2 - (x_3^2 - x_4^2) \frac{(x_3^2 + x_4^2 - x_1^2 - x_2^2)(1 + e)}{2} - 2(1 + e)x_1x_2(x_1x_3 + x_2x_4).$$

Then

$$L(X) = 2 - e \quad \forall X \in K^*_e.$$  [4.6]
Here we use the same notation for the gradient set of $v^e$ as in section 3.

When $x, y \in S^1$, a straightforward calculation yields

$$L(\nabla v^e(x) - \nabla v^e(y)) = (4 - 2\varepsilon)(1 - \langle x, y \rangle) - (\varepsilon + 1) \frac{\langle x, y \rangle(1 - \langle x, y \rangle) \varepsilon}{2}$$

$$+ (\varepsilon + 1)^2 \frac{\langle x, y \rangle}{2} (x_2y_1 - x_3y_2 - x_3y_4 + x_4y_3)^2.$$  

Write $\langle x, y \rangle = t$; when $t \geq 0$, we have

$$L(\nabla v^e(x) - \nabla v^e(y)) \geq (4 - 2\varepsilon - \frac{(\varepsilon + 1)^2}{2} \langle x, y \rangle(1 + \langle x, y \rangle))(1 - \langle x, y \rangle)$$

$$\geq (3 - 4\varepsilon - \varepsilon^2)(1 - \langle x, y \rangle);$$

and when $t < 0$,

$$L(\nabla v^e(x) - \nabla v^e(y)) \geq (4 - 2\varepsilon)(1 - \langle x, y \rangle) + \frac{(\varepsilon + 1)^2}{2} (x_2y_1 - x_3y_2 - x_3y_4 + x_4y_3)^2$$

$$\geq (4 - 2\varepsilon)(1 - \langle x, y \rangle) - \frac{(\varepsilon + 1)^2}{2} |x - y|^2$$

$$\geq (3 - 4\varepsilon - \varepsilon^2)(1 - \langle x, y \rangle).$$

Hence, if we take $0 \leq \varepsilon < \sqrt{7} - 2$, the conclusion follows by the fact that

$$-\nabla L(\nabla v^e(x))(\nabla v^e(y) - \nabla v^e(x)) = L(\nabla v^e(x) - \nabla v^e(y))$$

$$d_1(\varepsilon)|y - x|^2 \leq |\nabla v^e(x) - \nabla v^e(y)|^2 \leq d_2(\varepsilon)|y - x|^2.$$  

We then can follow the procedure in section 3 to finish the construction of $f^e$.

5. An Example of Nonuniqueness of Weak Solutions in $W^{1,2-\varepsilon}$

Let $\Omega$ be the unit ball in $\mathbb{R}^3$, we consider $w^e : \Omega \rightarrow \mathbb{R}^3$ given by

$$w^e(x) = \frac{x}{|x|^2}, \quad \frac{3}{2} < \varepsilon < 3. \quad [5.1]$$

Direct calculation shows that for $L(X) = -\text{Tr} \operatorname{cof}(X)$, we have

$$L(X) = 2\varepsilon - 3, \quad X \in K^e_1;$$

$$L(X - Y) = \frac{\varepsilon^2}{2} |X - Y|^2, \quad X, Y \in K^e_1;$$

where $K^e_1 = \{ \nabla w^e(x), x \in S^1 \}$. Then we can follow the same procedure to construct smooth, uniformly convex $f^e$ such that $w^e$ satisfies

$$\text{div} \nabla f^e(\nabla w^e) = 0 \quad [5.2]$$

in the sense of distributions. On the other hand, we know $u = x$ is the unique $W^{1,2}$ weak solution of Eq. 5.2 from general theory. Note that for our choice of $\varepsilon$, $w^e$ is in $W^{1,p}(\Omega, \mathbb{R}^3)$ for $1 < p < 3/\varepsilon$ but not in $W^{1,2}(\Omega, \mathbb{R}^3)$, which thus gives a counterexample to uniqueness of equations of type 1.2 in $W^{1,p}$ space.

We summarize what we have proved in Theorem 1 ($B^e_1$ denotes the unit ball in $\mathbb{R}^e$).

Theorem 1.

(i) Let $u^e : B^e_1 \rightarrow \mathbb{R}^m$ be given by Eq. 3.1, where $m = \lfloor n(n + 1)/2 \rfloor - 1$. Then for

$$0 \leq \varepsilon < \frac{\sqrt{3(n + 1)}}{n - 1} - 1,$$

there exists a smooth, uniformly convex function $f^e : M^{m \times n} \rightarrow \mathbb{R}$ such that $|D^2f^e| \leq c$ in $M^{m \times n}$ and

$$\text{div} \nabla f^e(\nabla u^e) = 0 \quad \text{in} \ \mathbb{R}^e.$$

(ii) Let $v^e : B^e_1 \rightarrow \mathbb{R}^3$ be given by Eq. 4.1. For $0 \leq \varepsilon < \sqrt{7} - 2$, there exists a smooth, uniformly convex function $f^e : M^{3 \times 4} \rightarrow \mathbb{R}$ such that $|D^2f^e| \leq c$ in $M^{3 \times 4}$ and
\[ \text{div} \nabla^* (\nabla v) = 0 \quad \text{in} \ \mathbb{R}^4. \]

(iii) Let \( w^\varepsilon : B_1^3 \to \mathbb{R}^3 \) be given by Eq. 5.1. For \( \frac{3}{2} < \varepsilon < 3 \), there exists a smooth, uniformly convex function \( f^\varepsilon : M^{3 \times 3} \to \mathbb{R} \) such that \( |D^2 f^\varepsilon| \leq c \) in \( M^{3 \times 3} \) and

\[ \text{div} \nabla^* (\nabla w^\varepsilon) = 0 \quad \text{in} \ \mathbb{R}^3. \]