Elliptic Yang–Mills equation

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We discuss some recent progress on the regularity theory of the elliptic Yang–Mills equation. We start with some basic properties of the elliptic Yang–Mills equation, such as Coulomb gauges, monotonicity, and curvature estimates. Next we discuss singularity of stationary Yang–Mills connections and compactness theorems on Yang–Mills connections with bounded L^2 norm of curvature. We also discuss in some detail self-dual solutions of the Yang–Mills equation and describe a compactification of their moduli space.

The Yang–Mills equation has played a fundamental role in our study of physics and geometry and topology in last few decades. Its regularity theory is crucial to our understanding of its mathematical applications. The aim of this note is to give a brief tour of recent progress on regularity theory of the Yang–Mills equation in a Euclidean space or more generally, a Riemannian manifold.

In the following, unless specified, we assume for simplicity that M is an open subset in $\mathbb{R}^n$ with the Euclidean metric. Let $\mathbf{G}$ be a subgroup in $\mathbf{SO}(r)$ and $\mathbf{g}$ be its Lie algebra. But I should emphasize that all our discussions here are valid for any differential manifold with a Riemannian metric and any compact Lie group G.

1. Yang–Mills Connections

First we recall that a connection on M with values in $\mathbf{g}$ is of the form

$$A = A_i dx_i, \quad A_i \in \mathbf{g}, \qquad [1.1]$$

where $x_1, \ldots, x_n$ are Euclidean coordinates. Its curvature can be computed as follows:

$$F_A = dA + A \wedge A = F_{ij} dx_i \wedge dx_j \qquad [1.2]$$

and

$$F_{ij} = \frac{1}{2}\left(\partial_i A_j - \partial_j A_i + [A_i, A_j]\right), \qquad [1.3]$$

where ∂_i denotes the ith partial derivative and $[A, B] = AB - BA$ is the Lie bracket of $\mathbf{g}$.

The Yang–Mills functional is defined on the space of connections and given by

$$\Upsilon(A) = \frac{1}{4\pi^2}\int_M |F_A|^2 dV_g, \qquad [1.4]$$

where $|F_A|^2 = -\Sigma_{i,j}\,\mathrm{tr}(F_{ij}F_{ij})$. The Yang–Mills equation is simply its Euler–Lagrange equation

$$\sum_{i=1}^{n} (\partial_i F_{ij} - [F_{ij}, A_i]) = 0, \quad \forall\, j. \qquad [1.5]$$

If we denote by D_A the differential operator $dB - [B, A]$ and D_A^* is its adjoint, then Eq. **1.5** can be written simply as $D_A^* F_A = 0$. On the other hand, as the curvature of a connection we have the second Bianchi identity $D_A F_A = 0$, that is,

$$\partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} = [A_k, F_{ij}] + [A_i, F_{jk}] + [A_j, F_{ki}], \quad \forall\, i, j, k. \qquad [1.6]$$

We will call A a Yang–Mills connection if it satisfies Eqs. **1.5** and **1.6**.

The gauge group $\mathcal{G}$ consists of all smooth maps form M into $\mathbf{G} \subset \mathbf{SO}(r)$. It acts on the space of connections by assigning A to $\sigma(A) = \sigma A \sigma^{-1} - \sigma d\sigma^{-1}$ for each $\sigma \in \mathcal{G}$. Clearly, the Yang–Mills functional is invariant under the action of $\mathcal{G}$, and so is the Yang–Mills equation. In particular, it implies that the Yang–Mills equation is not elliptic. However, it is elliptic modulo gauge transformations. To see it, we assume that A is the so-called Columbus gauge, that is $\Sigma_i\, \partial_i A_i = 0$, then the Yang–Mills equation reads

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Abbreviation: asd instantons, Ω-anti-self-dual instantons.

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$$\sum_{i=1}^{n} \partial_i^2 A_i + \text{ terms involving only lower-order derivatives of } A_\ell = 0.$$

Given any connection A, there is a gauge transformation $\sigma \in \mathcal{G}$ such that $\sigma(A)$ is in the Columbus gauge, that is

$$\sum_{i=1}^{n} \partial_i(\sigma A_i \sigma^{-1}) + (\partial_i \sigma)\sigma^{-1} = 0. \qquad [1.7]$$

The local solvability of this equation on gauge transformations has been shown by K. Uhlenbeck (1). It follows from

Theorem 1.1. (*From ref. 1*) *Let $A = A_i dx_i$ be any connection with $A_i \in L^p(B_1(p), \mathbf{g})$ for some $p \geq n/2$, where $B_1(p)$ is a unit ball in $\mathbb{R}^n$. Then there exists $\varepsilon(n) > 0$ and $c(n) > 0$ such that if $\|F_A\|_{n/2} \leq \varepsilon(n)$, where $\|\cdot\|_q$ denotes the L^q-norm in $B_1(p)$, then there is a gauge transformation σ satisfying Eq.* **1.7** *and $\|\sigma(A)\|_p \leq c(n)\|F_A\|_p$.*

In general, in the same way as we did above, one can introduce Yang–Mills connections for any vector bundle over any Riemannian manifold with structure group **G**. If **G** is $U(1)$, then a Yang–Mills connection is simply a purely imaginary valued 1-form whose curvature is a harmonic 2-form. So the theory of Yang–Mills connections is reduced to the Hodge theory for 1-forms.

2. Monotonicity and Its Consequences

Given any vector field X on M with compact support, we can integrate it to get a one-parameter group of diffeomorphisms ϕ_t: $M \mapsto M$. Put $A_t = \phi_t^*(A)$. Then $A_0 = A$ and A_t coincides with A near the boundary of M. If A is a smooth Yang–Mills connection, differentiating $\Upsilon(A_t)$ on t at $t = 0$, one can derive as Price did in ref. 2

$$\int_M \left(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^{n} F_{ij}F_{kj}\partial_i X^k\right) dV = 0, \qquad [2.1]$$

where $X = X^k \partial_k$. This is very important even though it is nothing but the first variation of Υ along X. Let us derive some of its consequences. Let $p \in M$ such that the ball $B_{\rho_0}(p)$ with radius ρ_0 and center p is contained inside M. Then taking X to be $\xi(r) r \partial_r$, where r is the distance from p and ξ is a cut-off function in $B_{\rho_0}(p)$, we can get the monotonicity formula of Price.

Theorem 2.1. (*From ref. 2*) *Let A be any Yang–Mills connection on M. Then for any $0 \leq \sigma \leq \tau \leq \rho_0$, we have*

$$\tau^{4-n}\int_{B_\tau(p)} |F_A|^2 dV - \sigma^{4-n}\int_{B_\sigma(p)} |F_A|^2 dV = 4\int_{B_\tau(p)\setminus B_\sigma(p)} r^{4-n} \sum_i |F_A(\partial_r, \partial_i)|^2 dV. \qquad [2.2]$$

In particular, $\rho^{4-n}\int_{B_\rho(p)} |F_A|^2 dV$ is nondecreasing with ρ.

An application of this monotonicity is the following curvature estimate, which was proved by K. Uhlenbeck [ref. 1; also see Nakajima (3)].

Theorem 2.2. *Let A be any Yang–Mills connection on U. Then there are $\varepsilon = \varepsilon(n) > 0$ and $C = C(n) > 0$, such that for any $B_\rho(p) \subset M$, we have*

$$|F_A|(p) \leq \frac{C}{\rho^2}\left(\rho^{4-n}\int_{B_\rho(p)} |F_A|^2 dV\right)^{1/2}, \qquad [2.3]$$

whenever $\rho^{4-n}\int_{B_\rho(p)} |F_A|^2 dV \leq \varepsilon$.

We refer the readers to ref. 3 (also ref. 4) for its proof. This curvature estimate implies that a Yang–Mills connection is almost flat whenever its normalized action in a neighborhood ball is sufficiently small.

We can associate a measure μ_A to each connection A as follows: For any continuous function f with compact support, we define

$$\int_M f\mu_A = \int_M f|F_A|^2 dV. \qquad [2.4]$$

We can simply write $\mu_A = |F_A|^2 dV$. By the monotonicity, we have is a nondecreasing function $\rho^{4-n}\mu_A(B_\rho(p))$.

Now we let $\{A_i\}$ be a sequence of Yang–Mills connections such that for each compact subset $K \subset M$, $\mu_i(K)$ are uniformly bounded, where μ_i is the measure associated to A_i. Then a subsequence $\{\mu_a\}$ of $\{\mu_i\}$ converges weakly to a measure μ. Because of the monotonicity for μ_i, one can easily show that $\rho^{4-n}\mu(B_\rho(p))$ is a nondecreasing function for each $p \in M$. Define the density function of μ by

$$\Theta_\mu(p) = \lim_{\rho \to 0} \rho^{4-n}\mu(B_\rho(p)). \qquad [2.5]$$

Because of the monotonicity for μ, this density Θ_μ is well defined, nonnegative, and upper-semi-continuous. It follows that the support S of Θ_μ is a locally closed subset of M such that the Hausdorff measure $\mathcal{H}^{n-4}(S \cap K)$ is finite for any compact subset K. Furthermore, it follows from *Theorem 2.2* that $\Theta_\mu(p) \geq \varepsilon$ for any $p \in S$ and the curvature of A_a is uniformly bounded on any compact subset in $M \backslash S$. Then, using *Theorem 2.2*, one can show the following theorem, which is due to Uhlenbeck.

Theorem 2.3. (*From ref. 5*) *Let A_a, μ_a, μ and S be as above. Then there are gauge transformations $\sigma_a \in \mathcal{G}$ such that by taking a subsequence if necessary, $\sigma_a(A_a)$ converges smoothly to a Yang–Mills connection A defined on $M \backslash S$. Moreover, $\mu_A \leq \mu$.*

By an admissible Yang–Mills connection, we mean a smooth Yang–Mills connection A defined outside a locally closed subset $S(A)$ in M, such that $\mathcal{H}^{n-4}(S(A) \cap K) < \infty$ and $\mu_A(K) < \infty$ for any compact subset $K \subset M$. Clearly, the limiting connection in *Theorem 2.3* is admissible. In fact, following Uhlenbeck (5), one can easily extend *Theorem 2.3* to any sequence of admissible Yang–Mills connections. We will assume that $S(A)$ is the singular set of an admissible Yang–Mills connection A. If $S(A) = \emptyset$, then A is smooth.

3. Removable Singularity Theorem

Let A be an admissible Yang–Mills connection with singular set $S(A)$. We say that A is stationary if Eq. **2.1** holds for any smooth vector field X with compact support. As we have shown in last section, any smooth Yang–Mills connection is stationary. However, not every admissible Yang–Mills connection is stationary.

Theorem 3.1. *Let A be a stationary admissible Yang–Mills connection and smooth on $M \backslash S$, where S is a closed subset in M and has locally finite $(n-4)$-dimensional Hausdorff measure. Then there is an $\varepsilon > 0$, which depends only on n, such that for any $B_\rho(p) \subset\in S(A)$, if*

$$\rho^{4-n} \int_{B_\rho(p)} |F_A|^2 dV < \varepsilon, \qquad [3.1]$$

then there is a gauge transformation σ near p such that $\sigma(A)$ extends to be a smooth connection near p.

When $n \leq 3$, $S(A)$ is empty. When $n = 4$, $S(A)$ consists of finitely many points and Eq. **2.1** holds for any admissible Yang–Mills connections. Hence, this theorem reduces to the removable singularity theorem of K. Uhlenbeck for Yang–Mills connections on 4-manifolds (1). When $n > 4$, this theorem was first proved in ref. 4 under certain conditions on A and was proved in ref. 6 for general cases.

Corollary 3.1. *Let A be a stationary admissible Yang–Mills connection. Then there is a gauge transformation σ such that $\sigma(A)$ is smooth outside a locally closed subset S' with vanishing $(n-4)$-dimensional Hausdorff measure, that is $\mathcal{H}^{n-4}(S') = 0$. If $n = 4$, then $\sigma(A)$ is actually smooth.*

We propose the following:

Conjecture 3.1. *Let A be a stationary admissible Yang–Mills connection, then there is a gauge transformation σ such that $\sigma(A)$ extends to be a smooth connection outside a locally closed subset with locally finite Hausdorff measure of dimension $n-5$.*

4. Structure of Blow-up Loci

Let $\{A_i\}$ be a sequence of smooth Yang–Mills connections such that its associated measures μ_i converge weakly to a measure μ. As before, we denote by Θ_μ the density and by S the support of μ. By *Theorem 2.3* and taking a subsequence if necessary, we may assume that there are gauge transformations σ_i such that $\sigma_i(A_i)$ converge to an admissible Yang–Mills connection A outside S.

Now we will examine the structure of S. Let μ_A be the measure associated to A. Define

$$S_b(\{A_i\}) = \overline{\left\{p \in M \mid \Theta_\mu(p) > 0, \lim_{r \to 0} r^{4-n} \int_{B_r(p)} |F_A|^2 dV = 0\right\}}. \qquad [4.1]$$

This set is called the blow-up locus of $\{A_i\}$. If no confusion occurs, we will simply write S_b for this blow-up locus. It is easy to see that $\mathcal{H}^{n-4}(S \backslash S_b) = 0$. The following proposition was proved in ref. 4. It gives the first regularity on the blow-up locus.

Proposition 4.1. *Let $\{A_i\}$ be the above sequence of Yang–Mills connections that converge to A. Then its blow-up locus S_b is $\mathcal{H}^{n-4}$-rectifiable; that is, for $\mathcal{H}^{n-4}$-a.e. p in S_b, there is a unique tangent space T_pS_b of S_b at p. Moreover, for any smooth function f with compact support, we have*

$$\int_M f d\mu = \int_M f d\mu_A + \int_{S_b} f \Theta_\mu d\mathcal{H}^{n-4}. \qquad [4.2]$$

Furthermore, there are constraints on the geometry of the blow-up loci.

Theorem 4.1. (*From ref. 4*) *For any vector field X with compact support in M, we have*

$$-\int_{S_b} \operatorname{div}_{S_b} X\Theta_\mu d\mathcal{H}^{n-4} = \int_M \left(|F_A|^2 \operatorname{div} X - 4 \sum_{i,j=1}^{n} F_{ij}F_{kj}\partial_i X^k \right) dV, \qquad [4.3]$$

where $\operatorname{div}_{S_b} X$ denotes the divergence of X along S_b and F_{ij} are the components of F_A.

Corollary 4.1. *If A is stationary, then S_b is stationary; that is, S_b has no boundary in M and its generalized mean curvature vanishes.*

I doubt that A is stationary in general, but it is stationary when A is anti-self-dual (cf. next section). If $A = 0$, then S_b is stationary and the curvature of A_i concentrates near a minimal variety of codimension 4. It leads to the question: *Let S be a minimal submanifold S of dimension $n - 4$ in general position; is S the limit of a sequence of Yang–Mills connections?*

We will call the above (A, S_b, Θ_μ) a generalized Yang–Mills connection. Two generalized Yang–Mills (A, S_b, Θ) and (A', S'_b, Θ') if A and A' are gauge equivalent on an open dense subset. The set of all generalized Yang–Mills connections modulo gauge transformations is precompact.

Theorem 4.1 can also be used to prove the existence of tangent cones for generalized Yang–Mills connections. Let A be a stationary admissible Yang–Mills connection with singular set $S(A)$. For any $\lambda > 0$ and $p \in S(A)$, we can define

$$A_\lambda(q) = \lambda \sum_i A_i(p + \lambda(q - p))dx_i\,,$$

where $A = \Sigma_i A_i dx_i$. Then there are sequences $\{\lambda(i)\}$ such that $\lim_{i\to\infty} \lambda(i) = 0$ and $A_{\lambda(i)}$ converge to a connection A^c outside S_c with $\mathcal{H}^{n-4}(S^c \cap B_R(0)) < \infty$ for any $R > 0$. Further, measures $|F_{A_i}|^2 dV$ converge weakly to a measure μ_c with density Θ_c. From *Theorem 4.1* follows

Corollary 4.2. *Let $A_{\lambda(i)}$, A_c, S_c, Θ_c be as above. Then we have that $\partial_r\Theta_c = 0$, $a \cdot S_c = S_c$ and $F_{A_c}(\partial_r, \cdot) = 0$.*

5. Anti-Self-Dual Instantons

Anti-self-dual instantons provide special solutions of the Yang–Mills equation. They are widely used in physics, geometry, and topology.

Let Ω be a closed differential form on M of degree $n - 4$. Let us introduce Ω-anti-self-dual instantons, or simply asd instantons if no possible confusion may occur. For simplicity, we assume that $\mathbf{G} = \mathbf{SU}(r)$. Recall that the Hodge operator $*$ on differential forms is defined by

$$*(dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(l)}) = \operatorname{sign}(\sigma) dx_{\sigma(l+1)} \wedge \cdots \wedge dx_{\sigma(n)}, \quad l = 1, \ldots, n,$$

where σ is any permutation of $\{1, \ldots, n\}$. We say that a $\mathbf{SU}(r)$-connection A is an Ω-anti-self-dual instanton if its curvature $F_A = \Sigma\, F_{ij} dx_i \wedge dx_j$ satisfies

$$*(F_A \wedge \Omega) = -F_A\,, \qquad [5.1]$$

or equivalently

$$\sum_{i,j} F_{ij} dx_i \wedge dx_j \wedge \Omega = -\sum_{i,j} F_{ij} * (dx_i \wedge dx_j).$$

Using the closedness of Ω and the second Bianchi identity, one can easily show that any asd instantons are Yang–Mills connections. An asd instanton is an absolute minimizer of the Yang–Mills functional Υ if Ω has conorm no more than one.

The following theorem shows advantages of using asd instantons. We call A an admissible asd instanton if it is an admissible Yang–Mills connection and anti-self-dual wherever it is well defined.

Theorem 5.1. (*From ref. 4*) *Assume that Ω is a parallel form of degree $n - 4$. Let A be an admissible Ω-anti-self-dual instanton on M. Then A is stationary.*

In particular, combining this theorem with the Removable Singularity Theorem, we see that if A is an admissible asd instanton, then there is a gauge transformation τ such that the singular set S of $\tau(A)$ is of $\mathcal{H}^{n-4}(S) = 0$. In fact, we propose

Conjecture 5.1. *If A is an admissible asd instanton, then there is a gauge transformation τ such that $\mathcal{H}^{n-6}(S(\tau(A)) \cap K) < \infty$ for any compact $K \subset M$.*

Now we assume that $\{A_i\}$ is a sequence of Ω-anti-self-dual instantons that converge to an admissible Ω-anti-self-dual instanton A (cf. *Theorem 2.3*), where Ω is a form on M of degree $n - 4$. Let $S_b \subset M$ be the blow-up locus of $\{A_i\}$ with the density Θ_μ. Note that μ_i is the measure associated to A_i and $\lim_{i\to\infty} \mu_i = \mu$.

For any admissible connection A', we can associate a current $C_2(A')$ as follows: For any smooth form φ with compact support in M, we define

$$C_2(A)(\varphi) = \frac{1}{8\pi^2} \int_M \operatorname{tr}(F_{A'} \wedge F_{A'}) \wedge \varphi. \qquad [5.2]$$

Clearly, if A' is smooth, it is nothing else but the current represented by the Chern–Weil form defining the second Chern class, so it is closed. In general, it was proved in ref. 4 that $C_2(A')$ is closed in M.

Since S_b is rectifiable (*Proposition 4.1*), we can also define a current $C_2(S_b, \Theta_\mu)$ by

$$C_2(S_b, \Theta_\mu)(\varphi) = \frac{1}{8\pi^2}\int_M (\varphi, \Omega)\Theta_\mu d\mathcal{H}^{n-4}. \qquad \textbf{[5.3]}$$

Theorem 5.2. (*From ref. 4*) *Let A_i, A, et al. be as above. Then $(1/8\pi^2)\Theta_\mu$ is integer-valued and S_b is calibrated by Ω; that is, for $\mathcal{H}^{n-4}$-a.e. $p \in S_b$ where T_pS_b exists, the restriction of Ω to T_pS_b coincides with the induced volume form. Moreover, we have*

$$\lim_{i\to\infty} C_2(A_i) = C_2(A) + C_2(S_b, \Theta_\mu). \qquad \textbf{[5.4]}$$

In particular, for any compact K, we have

$$\lim_{i\to\infty} \mu_i(K) = \mu_A(K) + \int_{S_b\cap K} \Theta_\mu d\mathcal{H}^{n-4}.$$

A simplified situation of *Theorem 5.2* can be described as follows: Let $\pi: \mathbb{R}^n \mapsto \mathbb{R}^4$ be an orthogonal projection and B be an asd instanton on $\mathbb{R}^4$. Then the pull-back $A = \pi^*B$ is Ω-asd if and only if $L = \pi^{-1}(0)$ is an Ω-calibrated subspace. This can be checked directly. As before, we ask if *an Ω-calibrated submanifold is the limit of a sequence of Ω-asd instantons*.

It is well known (cf. ref. 7) that if $|\Omega| \leq 1$, then any integral current calibrated by Ω is minimizing in its homology class. The corollary follows

Corollary 5.1. *Assume that $|\Omega| \leq 1$. Let S_b be the blow-up locus of a sequence of asd instantons A_i converging to A and Θ_μ be its associated density. Then $C_2(S_b, \Theta_\mu)$ is an area-minimizing integral current.*

The support S_b of $C_2(S_b, \Theta_\mu)$ may not be smooth. However, one can show that a dense open subset of S_b is smooth. Further, we do expect

Conjecture 5.2. *Let Ω be any closed differential form with $|\Omega| \leq 1$. Then Ω-calibrated integral current is supported on the closure N of a smooth manifold N_0 such that $N\backslash N_0$ is of codimension at least two.*

We end this section with an example. Assume that $n = 2m$. Fix an identification $\mathbb{R}^n = \mathbb{C}^m$. Let ω be given in complex coordinates $z_1, \ldots, z_m$ by

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^m dz_i \wedge d\bar{z}_i.$$

Put $\Omega = \omega^{m-2}/(m-2)!$. Then an Ω-asd instanton A is simply a Hermitian–Yang–Mills connection; that is, $F_A^{0,2} = 0$ and $F_A^{1,1}\cdot\omega = 0$, where $F_A^{k,l}$ is the (k, l)-part of F_A. Moreover, a subspace $L \subset \mathbb{R}^n$ of codimension 4 is Ω-calibrated if and only if L is a complex subspace in $\mathbb{C}^m$. Let S be the blow-up locus of a sequence of Hermitian–Yang–Mills connections and Θ be its associated density. Then $C_2(S, \Theta)$ is a closed integral current whose tangent spaces are complex subspaces. It follows from a result of J. King (8) that there are positive integers m_a and irreducible complex subvarieties V_a such that for any smooth φ with compact support in M,

$$C_2(S, \Theta)(\varphi) = \sum_a m_a \int_{V_a} \varphi.$$

It can be also proved that if A is an admissible asd instanton with respect to $\omega^{m-2}/(m-2)!$, then there is a gauge transformation τ such that $\tau(A)$ extends to be a smooth connection outside a complex subvariety of codimension greater than 2. For more details, see Tian and Yang (9).

6. Compactifying Moduli Spaces

In this section, I give an application. I will give a natural compactification of the moduli space of asd instantons.

Now we let M be a compact n-manifold with a Riemannian metric g and Ω be a closed form of degree $n-4$. Let E be a unitary vector bundle over M. Recall that $\mathfrak{M}_{\Omega,E}$ consists of all gauge equivalence classes of Ω-asd instantons of E over M. In general, $\mathfrak{M}_{\Omega,E}$ may not be compact. So we will compactify it.

A generalized Ω-asd instanton is made of an admissible Ω-asd instanton A of E, which extends to a smooth connection over $M\backslash S(A)$ for a closed subset $S(A)$ with $\mathcal{H}^{n-4}(S(A)) = 0$, and a closed integral current $C = C_2(S, \Theta)$ calibrated by Ω, such that cohomologically,

$$\mathrm{PD}[C_2(A)] + \mathrm{PD}[C_2(S, \Theta)] = C_2(E). \qquad \textbf{[6.1]}$$

where $C_2(E)$ denotes the second Chern class of E. Two generalized Ω-asd instantons (A, C), (A', C') are equivalent if and only if $C = C'$ and there is a gauge transformation σ on $M\backslash S(A) \cup S(A')$, such that $\sigma(A) = A'$ on $M\backslash S(A) \cup S(A')$. We denote by $[A, C]$ the gauge equivalence class of (A, C). We identify $[A, 0]$ with $[A]$ in $\mathcal{M}_{\Omega,E}$ if A extends to a smooth connection of

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E over M modulo a gauge transformation. We define $\overline{\mathfrak{M}}_{\Omega,E}$ to be set of all gauge equivalence classes of generalized Ω-asd instantons of E over M.

The topology of $\overline{\mathfrak{M}}_{\Omega,E}$ can be defined as follows: a sequence $[A_i, C_i]$ converges to $[A, C]$ in $\overline{\mathfrak{M}}_{\Omega,E}$ if and only if there are representatives (A_i, C_i) such that their associated currents $C_2(A_i, C_i)$ converge weakly to $C_2(A, C)$ as currents, where

$$C_2(A', C') = C_2(A') + C_2(S', \Theta'), \quad C' = (S', \Theta').$$

It is not hard to show that, by taking a subsequence if necessary, $\tau_i(A_i)$ converges to A outside $S(A)$ and the support of C for some gauge transformations τ_i.

Theorem 6.1. (*From ref. 4*) *For any M, G, Ω, and E as above, $\overline{\mathfrak{M}}_{\Omega,E}$ is compact with respect to this topology.*

When M is an m-dimensional compact Kähler manifold with Kähler form ω, Ω-asd instantons are Hermitian–Yang–Mills connections, where $\Omega = \omega^{m-2}/(m-2)!$. A generalized Ω-asd instanton consists of a holomorphic cycle together with a Hermitian–Yang–Mills connection of a reflexive sheaf. In particular, the compactification $\overline{\mathfrak{M}}_{\Omega,E}$ can be explicitly described (cf. ref. 9).

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