under a 180° rotation (e.g., ellipses, rectangles,. . .), and for any up–down symmetric configuration (e.g., an equilateral triangle with a vertical side).

Numerical Method

To obtain the emptying line for a given capillary geometry (Ω and ϕ), we must solve Eq. **S6** for each value of the contact angle θ. To do this, we find first the minimum of [**S7**] for a given value of the capillary length a, and compute its energy. The emptying line corresponds to the value of a for which this energy is zero. This is a deceptively difficult task for two main reasons:

i) A formal minimization of the functional [**S7**] yields the differential equation of a 2D drop inside the capillary cross-section under gravity, with the anticipated boundary condition that the drop must touch the capillary walls with the contact angle θ. In Cartesian coordinates, this equation reads

$$\frac{\eta''(x)}{\left(1+\eta'(x)^2\right)^{3/2}} = \pm\,\frac{1}{a^2}(x\sin\phi+\eta\cos\phi)+\lambda, \qquad \textbf{[S8]}$$

where the sign refers to the liquid being below (+) or above (−) the gas at the point $(x,\eta(x))$. In general, the presence of gravity precludes any analytical solution, and numerical methods must be adopted. Additionally, this equation is known to exhibit multiple solutions (26). The emptying line corresponds to the smallest value of the gravity acceleration (the largest value of a) for which Eq. **S6** is verified. If the energies of two different solutions vanish for the same value of a, this means that there exist two equivalent (yet different) emptying mechanisms for the capillary.

ii) In addition to their shape, the location of the drops (the starting point of integration) is not known a priori, which increases the complexity of the problem.

To solve these problems, we parametrize the perimeter of the capillary cross-section $\partial\Omega$ with a variable t. Each value of the parameter $t\in[-\pi,\pi)$ refers to a unique point in the perimeter $(x_{\partial\Omega}(t),y_{\partial\Omega}(t))$. Next, we define a function $\Psi(t)$ as the value of [**S7**] that is obtained from integration of the drop equation starting at the point $(x_{\partial\Omega}(t),y_{\partial\Omega}(t))$ making the correct angle θ with the capillary wall. As the initial point of integration and the derivative are known, the integration can be performed, akin to a shooting method, ending when the drop reaches another point on the wall. In general, the contact angle condition is not verified at this second point of contact. However, because the parameter t spans the entire perimeter, all drops satisfying the contact angle condition at both points of contact are necessarily included, and occur at specific values of t. These drops correspond to the stationary points of the functional $E^{\,2D}[\eta]$, and appear as extrema of the function $\Psi(t)$. Note, however, that not every extremum of $\Psi(t)$ corresponds necessarily to a stationary point of $E^{\,2D}[\eta]$, with the exception of the global minimum of $\Psi(t)$, which is a guaranteed minimizer of $E^{\,2D}[\eta]$ and, indeed, its global minimum.

What makes the phenomena of capillary emptying so rich is that solutions to the Young–Laplace equation may disappear abruptly, as pointed out by Finn (26). This means that the function $\Psi(t)$ is discontinuous, and displays an extraordinary sensitivity to the capillary geometry and to changes of the variables θ and a. This is illustrated in Fig. S3, where the two global minima of $\Psi(t)$ represent different emptying mechanisms.

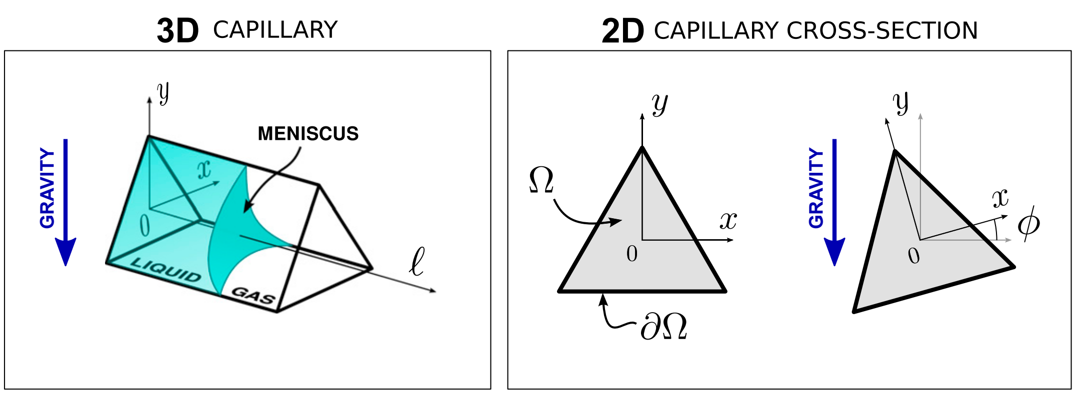

Fig. S1. Schematic drawing of a capillary and its cross-section, indicating the choice of coordinates. The variable ϕ represents the rotation angle of the capillary around its longitudinal axis.

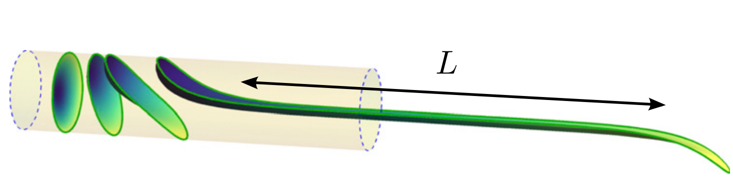

Fig. S2. Schematic drawing of menisci in a horizontal capillary approaching the emptying line as the capillary length a is decreased progressively (from left to right). When the emptying line is reached (for a particular value of a), the length L diverges. Although the liquid volume is conserved, the menisci have been shifted along the capillary to facilitate representation.

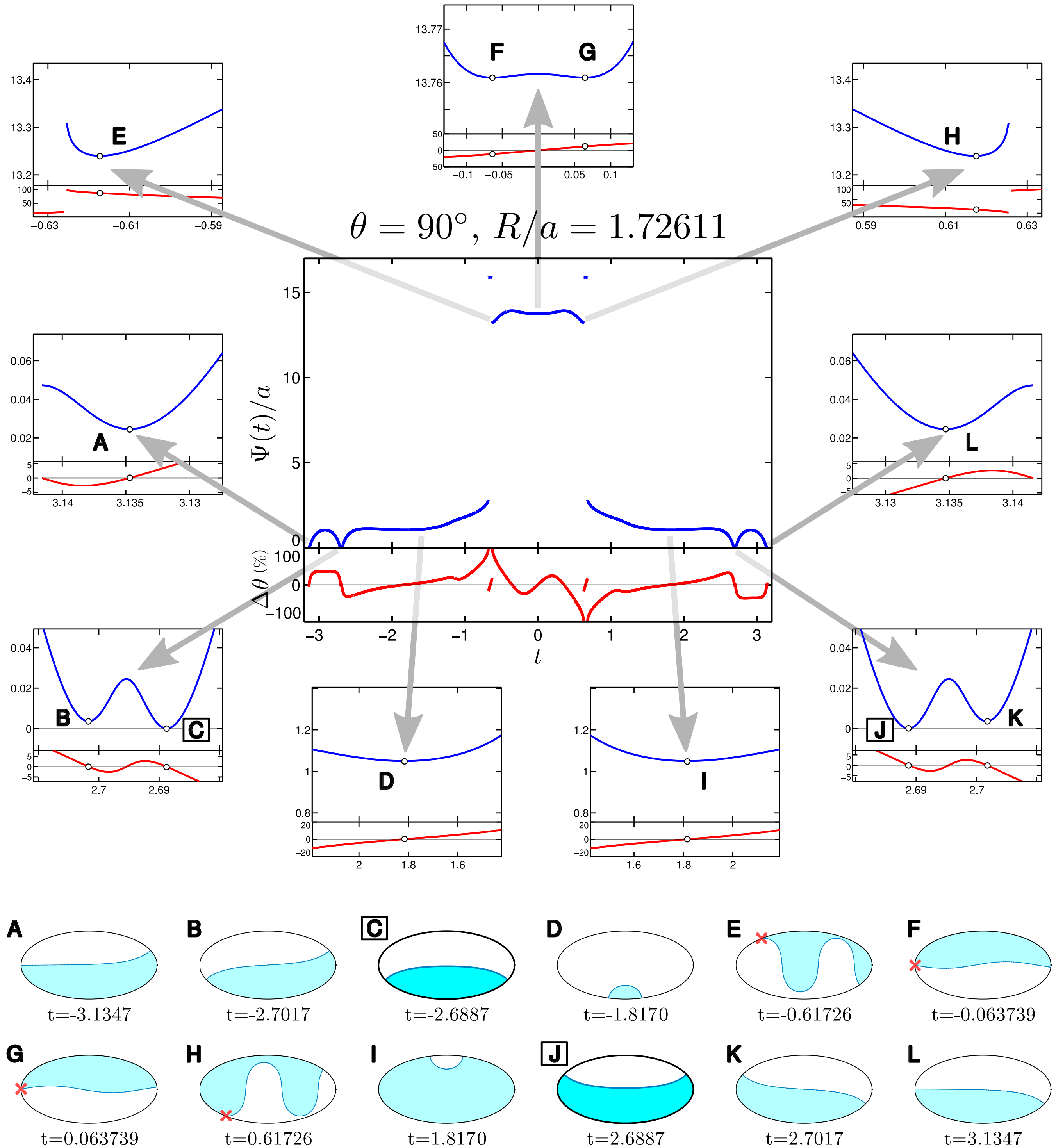

Fig. S3. Blue line: Function $\Psi(t)$ for the point ($\theta = 90°$, $R = 1.72611a$) on the emptying line of a horizontal capillary with an elliptical cross-section ($x^2_{\partial\Omega} + (1/4)y^2_{\partial\Omega} = R^2$) oriented at an angle $\phi = 90°$. Red line: Relative error in the contact angle condition $\Delta\theta$ at the second point of contact [check the definition of $\Psi(t)$ in the main text]. This error vanishes at all stationary points of the functional [**S7**]. Blowups: Details of $\Psi(t)$ and $\Delta\theta$ for all minima of the function $\Psi(t)$. *A*–*L*: Drop shapes corresponding to each of the minima of $\Psi(t)$. Drops *C* and *J* correspond to the absolute minima of $\Psi(t)$ and, hence, of the functional [**S7**]; as $\Psi(t)$ vanishes at these two minima, they verify Eq. **S6** and, therefore, represent equivalent (yet different) cross-sections of the (infinitely long) meniscus at capillary emptying. Drops *A*, *B*, *D*, *I*, *K*, and *L* are stationary points of the functional [**S7**]; they do not represent emptying mechanisms, as their value of Ψ is positive. Drops *E*–*H* are minima of $\Psi(t)$ but not stationary points of [**S7**], because the second point of contact of these drops with the capillary walls (red crosses) does not display the correct contact angle.