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Supplementary Information for

Two-level masers as heat-to-work converters

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Supporting Information Text

Derivation of the Raman Hamiltonian. Let us denote the hot-bath states with frequencies $\omega_k \simeq \omega_h$ and wave vectors k by their occupation numbers n_k and the signal-mode with occupation n_s by $|n_s\rangle$. If we have $|g, n_k, n_s\rangle = |\Psi_i\rangle$ as the initial and $|e, n_k - 1, n_s + 1\rangle = |\Psi_f\rangle$ as the final states, we will have experienced a Raman process (Figs. S2). According to time-dependent perturbation theory, the probability amplitude for a transition $|\Psi_i\rangle$ to $|\Psi_f\rangle$ after a time t is given by

$$\begin{aligned}\mathcal{A}_{i\to f} &= \langle\Psi_f|\left(1 + \frac{1}{i\hbar}\int_0^t dt_1 V(t_1) + \frac{1}{(i\hbar)^2}\int_0^t dt_1\int_0^{t_1} dt_2 V(t_1)V(t_2) + ...\right)|\Psi_i\rangle,\\ &\simeq -\frac{1}{\hbar^2}\int_0^t dt_1\int_0^{t'} dt_2\langle\Psi_f|V(t_1)V(t_2)|\Psi_i\rangle,\end{aligned} \quad [1]$$

since $\langle\Psi_f|\Psi_i\rangle = \langle\Psi_f|V|\Psi_i\rangle = 0$. We keep only the rotating-wave term and denote a virtual level that enables the Raman transition by $|u\rangle$ and its energy by ϵ_u, and the corresponding dipolar couplings by $g_{ug}^{(\omega_k)}$ and $g_{ue}^{(\nu)}$ respectively. The second-order probability amplitude of the two-photon transition $|\Psi_i\rangle = |g, n_k, n_s\rangle \Rightarrow |u, n_k - 1, n_s\rangle \Rightarrow |\Psi_f\rangle = |e, n_k - 1, n_s + 1\rangle$ is then

$$-\mathcal{A}_{i\to f} \simeq \sum_k \int_0^t dt_1\int_0^{t_1} dt_2 g_{eu}^{(\nu)} g_{ug}^{(\omega_k)}\langle n_k - 1, n_s + 1|e^{it_1(\epsilon_e-\epsilon_u)}\mathsf{b}^\dagger e^{i\nu t_1}e^{it_2(\epsilon_u-\epsilon_g)}a_k e^{-i\omega_k t_2}|n_k, n_s\rangle. \quad [2]$$

Assuming that the level energies satisfy $\epsilon_u > \epsilon_e > \epsilon_g$ and $0 < \nu < \omega_k$, we take the frequencies to be non-resonant (in the one-photon sense). Thus we evaluate the amplitude of the rotating term to be

$$-\mathcal{A}_{i\to f} = \sum_k g_{eu}^{(\nu)} g_{ug}^{(\omega_k)}\frac{-i\sqrt{n_k}\sqrt{n_s+1}}{i(\epsilon_u - \epsilon_g - \omega_k)}\left\{\frac{e^{it[(\epsilon_e-\epsilon_g)-(\omega_k-\nu)]}-1}{(\epsilon_e-\epsilon_g)-(\omega_k-\nu)} - \frac{e^{it(\epsilon_e-\epsilon_u+\nu)}-1}{\epsilon_e-\epsilon_u+\nu}\right\}, \quad [3]$$

which, in the limit $t \to \infty$ and upon neglecting the principle-value terms, reduces to

$$-\mathcal{A}_{i\to f} = \sum_k g_{eu}^{(\nu)} g_{ug}^{(\omega_k)}(-2\pi i)\frac{\delta[(\epsilon_e-\epsilon_g)-(\omega_k-\nu)]}{(\epsilon_u-\epsilon_g-\omega_k)}\sqrt{n_k}\sqrt{n_s+1}. \quad [4]$$

The Raman Hamiltonian should yield the same transition amplitude from first-order perturbation theory

$$\mathcal{A}_{i\to f} = \langle\Psi_f|\left(1 + \frac{1}{i\hbar}\int_0^t dt_1 V_h(t_1)\right)|\Psi_i\rangle. \quad [5]$$

Therefore, the Raman Hamiltonian is

$$V_h(t) = 2\pi\hbar\sum_k \frac{g_{eu}^{(\nu)} g_{ug}^{(\omega_k)}}{\epsilon_u-\epsilon_g-\omega_k}\mathsf{b}^\dagger a_k e^{-i(\omega_k-\nu)t}|e\rangle\langle g|e^{i(\epsilon_e-\epsilon_g)t} + \text{H.c.}, \quad [6]$$

which is equivalent to

$$V_h(t) \quad = \quad \hbar\sum_k g_k^{(h)}\left(|e\rangle\langle g|\mathsf{b}^\dagger a_k^{(h)} e^{-i[\omega_k-(\nu+\omega_0)]t} + \text{H.c.}\right), \quad [7]$$

with the coupling constant

$$g_k^{(h)} = 2\pi\frac{g_{eu}^{(\nu)} g_{ug}^{(\omega_k)}}{\epsilon_u-\epsilon_g-\omega_h}, \quad [8]$$

taken to be real and $\omega_k \approx \omega_h$, approximated near the two-photon Raman resonance.

Our next goal is to obtain the master equation of motion for the density operator ρ of the joint (WM+signal) system coupled to the hot bath:

$$\dot\rho^{(h)} = -\frac{i}{\hbar}\text{Tr}_h[V(t), \rho(t_0)\otimes\rho_h(t_0)] - \frac{1}{\hbar^2}\text{Tr}_h\int_{t_0}^t [V(t), [V(t'), \rho(t')\otimes\rho_h(t_0)]dt'. \quad [9]$$

Inserting $V(t)$ into Eq. (9), we recall that $\langle a_k\rangle = \langle a_k^\dagger\rangle = 0$ and $\langle a_k a_{k'}\rangle = \langle a_k^\dagger a_{k'}^\dagger\rangle = 0$, where $\langle A\rangle = \text{Tr}_h[\rho_h(t_0)A]$ stands for tracing over the hot bath and we omit the superscript (h). Next we note that

$$\langle a_k^\dagger a_{k'}\rangle = \bar n_k\delta_{kk'}; \quad \langle a_k a_{k'}^\dagger\rangle = (\bar n_k + 1)\delta_{kk'}. \quad [10]$$

Then sum over k can be replaced by an integral as follows:

$$\sum_k \to \frac{\mathcal{V}}{\pi^2}\int_0^\infty d^3k, \quad [11]$$

where $\mathcal{V}$ is the hot-bath mode volume. Hereafter we neglect all memory effects (i.e., we adopt the Markov approximation) and assume that $\rho(t')$ is slowly varying. We can thus extend the integration over t' to ∞ and use

$$\int d^3k \int_{t_0}^{\infty} dt' e^{i(\omega - ck)(t-t')} \simeq L(\omega - \omega_h), \tag{12}$$

where $L(\omega - \omega_h)$ is a Lorentzian shape whose (narrow) width is determined by the cavity-mirror finesse, and

$$\frac{\omega_0 + \nu}{c} = \frac{\omega_h}{c}. \tag{13}$$

We then obtain from Eq. (9)

$$\begin{aligned}\dot{\rho}^{(h)} &= \gamma_h(\bar{n}_h + 1)([S\rho, S^\dagger] + [S, \rho S^\dagger]) \\ &+\gamma_h \bar{n}_h([S^\dagger \rho, S] + [S^\dagger, \rho S]),\end{aligned} \tag{14}$$

where

$$S = \mathsf{b}|g\rangle\langle e| \equiv \mathsf{b}\sigma_-, \tag{15}$$

and

$$\gamma_h = \frac{\mathcal{V}\omega_h^2 g_k^2}{\pi c^3}, \tag{16}$$

is the decay rate into the hot bath calculated for the Raman coupling squared g_k^2. The equation of motion (14) is identical to the standard master equation for a two-level system coupled to a thermal reservoir (1), which in our case governs the interaction of the WM with the cold bath:

$$\begin{aligned}\dot{\rho}^{(c)} &= \gamma_c(\bar{n}_c + 1)([\sigma_-\rho, \sigma_+] + [\sigma_+, \rho\sigma_+]) \\ &+\gamma_c \bar{n}_c([\sigma_+\rho, \sigma_-] + [\sigma_+, \rho\sigma_-]).\end{aligned} \tag{17}$$

via the Hamiltonian

$$V_c(t) \quad = \quad \hbar \sum_k g_k^{(c)} \left(|e\rangle\langle g| a_k^{(c)} e^{-i[\omega_k - \omega_0]t} + \text{H.c.} \right), \tag{18}$$

where the summation is over bath wavevectors k, and $g_k^{(c)}$ is the coupling constants, taken to be real. However, in the case of the hot bath, the transition operator $S = \mathsf{b}|g\rangle\langle e|$ involves both the WM and the signal. Futhermore, the hot bath coupling constant, arising from Eqs. (8) and (16), is typically much weaker than its cold bath counterpart. Taken together, master equations (14) and (17) govern the dynamics of the WM+signal system.

Evolution of the signal state. Since the $|g\rangle \leftrightarrow |e\rangle$ transition is primarily caused by the cold-bath transition at a rate $\gamma_c \gg \gamma_h$, we can assume that the WM attains its steady state in thermal resonance with the cold bath, so Eq. (1) of the main text holds. Upon tracing out the WM, we obtain the reduced (Lindblad) master equation for the signal-mode density matrix $\rho_s = \text{Tr}_W(\rho)$

$$\begin{aligned}\dot{\rho}_s &= \gamma_h(\bar{n}_h + 1)\rho_{ee}([\mathsf{b}\rho_s, \mathsf{b}^\dagger] + [\mathsf{b}, \rho_s \mathsf{b}^\dagger]) \\ &+\gamma_h \bar{n}_h \rho_{gg}([\mathsf{b}^\dagger \rho_s, \mathsf{b}] + [\mathsf{b}^\dagger, \rho_s \mathsf{b}]),\end{aligned} \tag{19}$$

and therefore

$$\dot{\bar{n}}_s = -2\gamma_h \left[\rho_{ee}\bar{n}_s(\bar{n}_h + 1) - \rho_{gg}\bar{n}_h(\bar{n}_s + 1)\right]. \tag{20}$$

In the semiclassical limit, setting $\bar{n}_s = I_s$ and ignoring spontaneous emission into the hot bath, we get Eqs. (3) and (4) of the main text.

To treat the signal quantum-mechanically, we use the Glauber-Sudarshan decomposition of the signal state:

$$\rho_s = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|, \tag{21}$$

where $P(\alpha)$ is the Glauber-Sudarshan P quasiprobability distribution. We then obtain the Fokker-Planck (FP) equation for this distribution (1–3)

$$\frac{\partial P}{\partial t} = -\frac{G}{2}\left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \alpha *}\right) P + D\frac{\partial^2 P}{\partial \alpha \partial \alpha *}, \tag{22}$$

with

$$G = \frac{2\gamma_h \bar{n}_h(\bar{n}_c + 1) - 2\gamma_h(\bar{n}_h + 1)\bar{n}_c}{2\bar{n}_c + 1} = \frac{2\gamma_h(\bar{n}_h - \bar{n}_c)}{2\bar{n}_c + 1}; \tag{23}$$

$$D = \frac{2\gamma_h \bar{n}_h(\bar{n}_c + 1)}{2\bar{n}_c + 1}. \tag{24}$$

Here G is the effective gain rate in the amplification regime [which is identical to the semiclassical one (3) of the main text] and D is the diffusion rate. For a coherent state input, this corresponds (1) to the P function behaving in accordance with Eq. (7) of the main text with $\sigma^2(t) = \frac{D}{G}(e^{Gt} - 1)$.

Implementation of the displacement operator. The beam splitter will enact the following transformation:

$$\begin{aligned} a_{\rm LO} &\to r a_{\rm LO} + \tau a; \\ a &\to r a - \tau a_{\rm LO}. \end{aligned} \tag{25}$$

Here $a_{\rm LO}$ is the amplitude operator of the local oscillator field (which can be treated as a c-number because of its high magnitude compared to $\alpha_0 e^{Gt/2}$), and $\tau \ll 1$ and r are the beam splitter transmissivity and reflectivity, respectively. By setting $\tau a_{\rm LO} = r\alpha_0 e^{Gt/2}$, we obtain a state of zero amplitude in the reflected channel of the signal. The energy of the local oscillator, on the other hand, will increase by the coherent-component energy according to

$$\hbar\nu a_{\rm LO}^2 \to \hbar\nu[r a_{\rm LO} + \tau\alpha_0 e^{Gt/2}]^2 \tag{26}$$

$$\approx \hbar\nu[(1-\tau^2/2)a_{\rm LO} + \tau^2 a_{\rm LO}]^2 \tag{27}$$

$$\approx \hbar\nu a_{\rm LO}^2(1+\tau^2) \tag{28}$$

$$\approx \hbar\nu a_{\rm LO}^2 + [\alpha_0 e^{Gt/2}]^2, \tag{29}$$

where we have approximated

$$r = \sqrt{1-\tau^2} \approx 1 - \tau^2/2. \tag{30}$$

Derivation of the maser efficiency. The efficiency can be computed as the ratio (4)

$$\eta = \frac{\dot{\mathcal{W}}_s}{\dot{Q}_h}, \tag{31}$$

where $\dot{\mathcal{W}}_s = \hbar\nu G|\alpha_0|^2 e^{Gt}$ and heat flux from the hot bath is given by $\dot{Q}_h = Tr_{\rm Ws}(\dot{\rho}^{(h)} H_{\rm WM+s}) = \hbar\omega_h[G\langle \mathsf{b}^\dagger \mathsf{b}\rangle + D]$.

Using Eq. (9) of the main text, $\langle H_s(t)\rangle = \hbar\nu\langle \mathsf{b}^\dagger \mathsf{b}\rangle$, the expression for the efficiency can be simplified as

$$\eta = \frac{\nu}{\omega_h}\frac{|\alpha_0|^2}{|\alpha_0|^2 + \frac{D}{G}}. \tag{32}$$

Equation (32) can be further rewritten as:

$$\eta = \eta_{SSD}\frac{\hbar\nu|\alpha_0|^2 e^{Gt} - \langle H_s(0)\rangle}{\langle H_s(t)\rangle - \langle H_s(0)\rangle} = \frac{\nu}{\omega_h}\frac{|\alpha_0|^2}{|\alpha_0|^2 + \frac{\bar{n}_h(\bar{n}_c+1)}{\bar{n}_h - \bar{n}_c}}, \tag{33}$$

where we take into account that the first term of the numerator Eq. (33) corresponds to extractable work and the ergotropy of the initial coherent state is $\langle H_s(0)\rangle = \hbar\nu|\alpha_0|^2$.

Under injected operation with independent coherent states as mentioned in the main text, the mean efficiency follows from Eq. (33) as

$$\langle\eta\rangle = \int \frac{\nu}{\omega_h}\frac{|\alpha_0|^2}{|\alpha_0|^2 + \frac{\bar{n}_h(\bar{n}_c+1)}{\bar{n}_h - \bar{n}_c}} p(\alpha_0)\mathrm{d}^2\alpha_0. \tag{34}$$

Going to polar coordinates, one then obtains

$$\langle\eta\rangle = \int \frac{\nu}{\omega_h}\frac{|\alpha_0|^2}{|\alpha_0|^2 + \frac{\bar{n}_h(\bar{n}_c+1)}{\bar{n}_h - \bar{n}_c}} |\alpha_0|\mathrm{d}|\alpha_0| \int p(\alpha_0) d\theta_0, \tag{35}$$

which reduces to Eq. (11) of the main text upon defining the *phase averaged* initial distribution $p(|\alpha_0|) = \int p(\alpha_0)d\theta_0$.

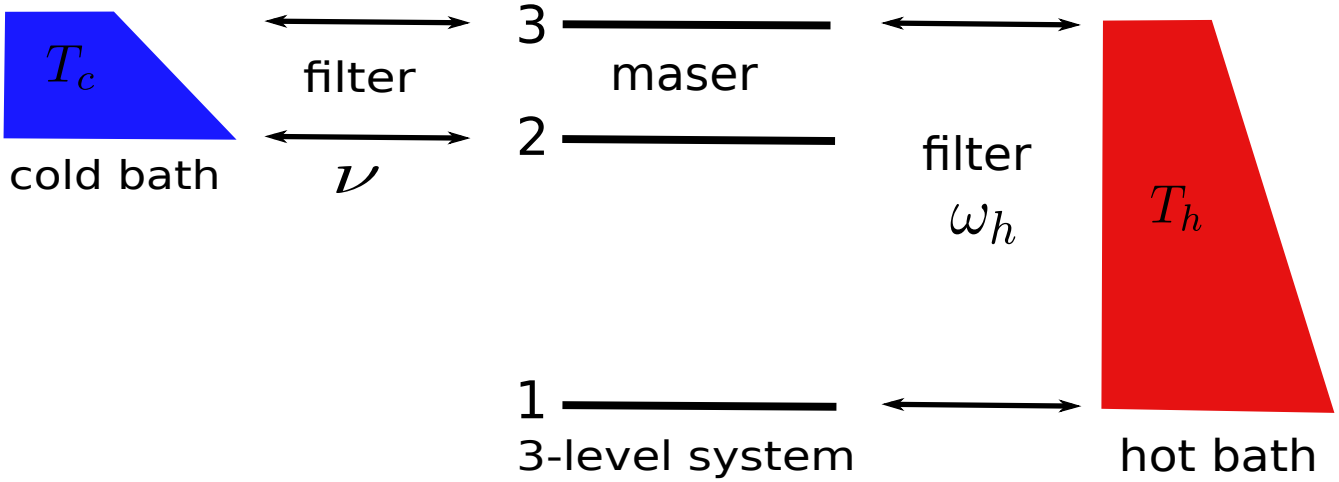

Fig. S1. Schematic layout of the canonical three-level Scovil-Schulz-DuBois maser scheme. This scheme requires bath-induced population inversion between levels 3 and 2 in order to amplify a maser signal that is resonant with the $3-2$ transition. This scheme is principally different from the present two-level scheme in Fig. 1b, where no such population inversion exists.

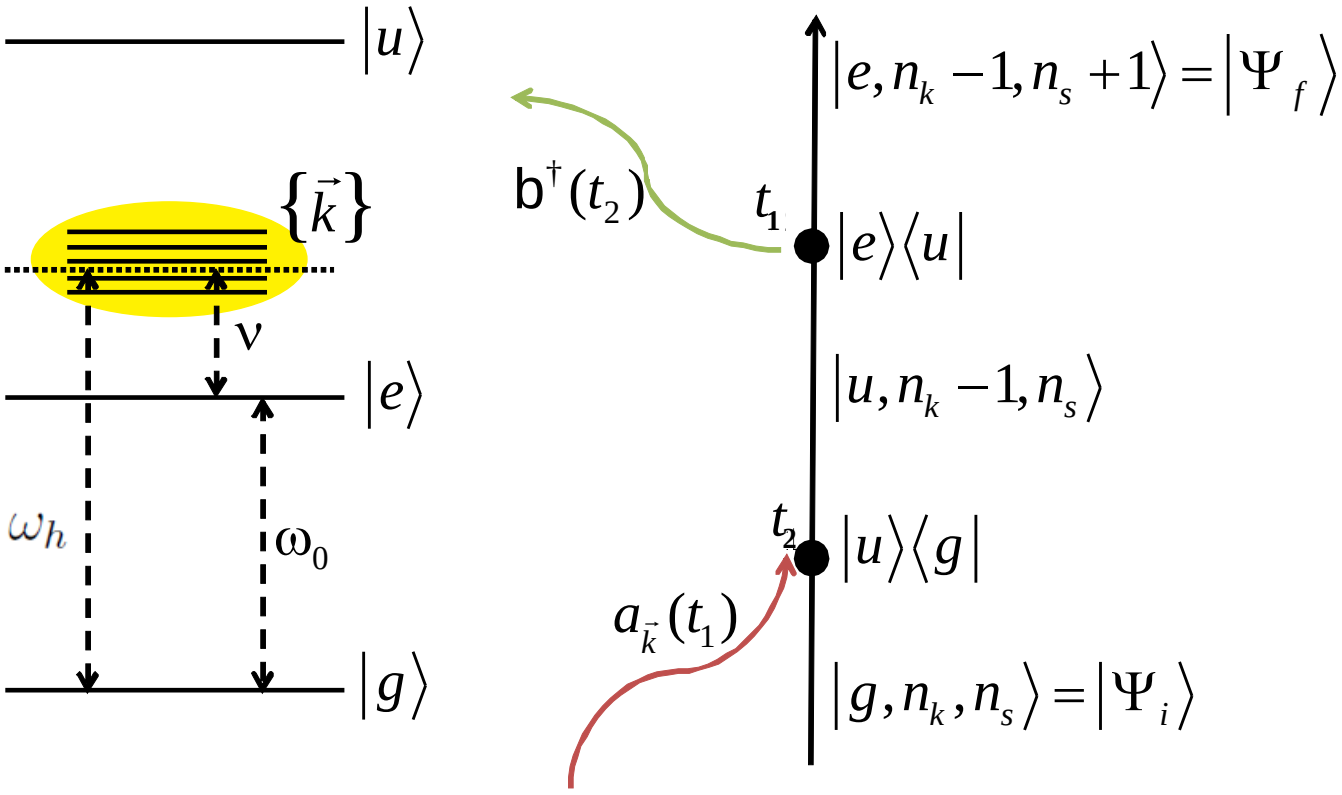

Fig. S2. Illustration to Raman-coupling Hamiltonian

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