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Conformally Flat Manifolds

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ABSTRACT This note describes some new examples of conformally flat manifolds, as a step toward a classification of such manifolds up to conformal equivalence.

A Riemannian manifold M^n is called *conformally flat* if it can be covered by neighborhoods $\{U_\alpha\}$ such that there exists a conformal map $\varphi_\alpha: U_\alpha \to \mathbf{R}^n$. From the point of view of conformal geometry, conformally flat manifolds are the "core" manifolds. Primary examples of conformally flat manifolds are manifolds of constant curvature. Recall also that every 2-dimensional Riemannian manifold is conformally flat. This follows from the existence of isothermal co-ordinates, and this fact is essentially equivalent to the existence of underlying complex structure if the manifold is orientable. In this way a conformally flat manifold may be considered as a generalization of a Riemann surface and one can expect analogies between the theory of Riemann surfaces, complex manifolds, and conformally flat manifolds.

The notion of conformal flatness is one of the most primitive concepts in differential geometry. In spite of this fact, excluding the case of a Riemann surface, most of the work on conformally flat manifolds has been of local character. An exception is the work of Kuiper (3), who has notably extended the idea of conformal development. One can pose the following basic and very broad problem.

PROBLEM. *Classify conformally flat manifolds up to conformal equivalence.*

Considering the complexity of the theory of Riemann surfaces and that of space forms, it is obvious that the problem is extremely complicated. To start with, it is important to have some classes of examples. It may be somewhat surprising that all examples known so far are essentially the manifolds of constant curvature and certain of their products. The purpose of this note is to describe some new examples. The details will be published elsewhere.

Conformal Surgery. I shall first describe a process of surgery on conformally flat manifolds. This shows in particular that the class of conformally flat manifolds is very rich. Recall that surgery on M^n involves imbedding $\iota: S^p \times D^{q+1} \to M^n, p + q + 1 = n$, removing the interior of $\iota(S^p \times D^{q+1})$, and smoothly glueing $D^{p+1} \times S^q$ along the boundary. A *standard imbedding* of $S^p \times D^{q+1}$ in $\mathbf{R}^n$, $p + q + 1 = n$, is the normal disc bundle of a standard S^p lying in a $(p + 1)$-plane in $\mathbf{R}^n$. An imbedding $\iota: S^p \times D^{q+1} \to M^n$, where M^n is conformally flat and $p + q + 1 = n$ is called *admissible* if there exists an isotopy ι_t of imbeddings $0 \leq t \leq 1$ such that $\iota_0 = \iota$, and such that there exists a neighborhood U of $\iota_1(S^p \times D^{q+1})$ and a conformal map $\varphi: U \to \mathbf{R}^n$ so that $\varphi \circ \iota_1$ is a standard imbedding. We call a surgery on M^n *admissible* if the corresponding imbedding $\iota: S^p \times D^{q+1} \to M^n$ is admissible. We have the following

THEOREM 1. *Let M^n be a conformally flat manifold, and $\tilde{M}^n$ a manifold obtained by performing admissible surgery on M^n. Then $\tilde{M}^n$ admits a conformally flat structure.*

We also have a similar

THEOREM 2. *Let M^n, N^n be conformally flat manifolds. Then their connected sum admits a conformally flat structure.*

*Hypersurfaces**. I have obtained a satisfactory classification of conformally flat hypersurfaces in $\mathbf{R}^{n+1}$, $n \geq 4$. A typical result is the following:

THEOREM 3. *Let M^n be a compact conformally flat manifold $n \geq 4$ and $\iota: M^n \to \mathbf{R}^{n+1}$ a conformal immersion. Suppose that both the metric and immersion of M are analytic. Then M^n is diffeomorphic to S^n or $S^{n-1} \times S^1$.*

The case when the metric and immersion are only assumed to be C^∞ is more interesting. The basic local proposition is the following:

PROPOSITION (Cartan, Schouten). *A hypersurface in $\mathbf{R}^{n+1}$, $n \geq 4$, is conformally flat iff at least (n-1) eigenvalues of the second fundamental form are equal at each point.*

Due to this proposition locally there turn out to be only four possibilities:

(*i*) *totally umbilic*—hence of constant curvature;

(*ii*) *flat* (but not necessarily totally umbilic);

(*iii*) *a surface of revolution:* Let $[t,x^1,\ldots,x^n]$ be orthonormal co-ordinates in $\mathbf{R}^{n+1}$. Choose a curve in the $[t,x^1]$ plane lying in the half plane $x_1 > 0$. Revolving this curve around the t-axis, we obtained a surface of revolution.

(*iv*) *a tube:* Choose any smooth curve in $\mathbf{R}^{n+1}$ and take its normal sphere bundle where the spheres are of a sufficiently small fixed radius. By definition the total space of the bundle is a tube. Roughly speaking a general conformally flat hypersurface is obtained by smooth glueing of these four possibilities. However, arbitrary glueing is not possible. For example a tube does not contain an umbilic so cannot be glued to a totally umbilic piece.

* It came to my notice when I had already finished this work that Cartan (1) and Schouten (4) also attempted classification of conformally flat hypersurfaces. However, both authors have obtained part of the local results only, and determination of global form of, say, a complete hypersurface was not considered by them.

A notable consequence of this classification is

THEOREM 4. *Let M^n be a conformaly flat manifold $n \geq 4$ and $\iota: M^n \to \mathbf{R}^{n+1}$ a conformal immersion as a complete hypersurface. Then $\pi_1(M^n)$ is free.*

COROLLARY. *Let M^n be a compact space form (i.e., of constant curvature) $n \geq 4$. If $M^n \not\approx S^n$ then M^n cannot be conformally immersed in R^{n+1}.*

In their recent work, Chern and Simons (2) have obtained necessary conditions for conformal immersions of Riemannian manifolds. These conditions are expressed in terms of secondary characteristic classes. The above corollary is not a consequence of these conditions, e.g., for a flat torus Chern–Simons obstructions vanish but it cannot be conformally immersed as a hypersurface. On the other hand, when the method of Chern and Simons works, it will in general give better results as it applies also for conformal immersions in arbitrary codimension.

The following are related results requiring a somewhat different technique.

THEOREM 5. *(conformal rigidity of a cylinder). Let $M^n = S^1 \times \mathbf{R}^{n-1}$ (product metric) $n \geq 4$ and $\iota: M^n \to \mathbf{R}^{n+1}$ an analytic, conformal imbedding such that $\iota(M^n)$ is complete. Then $\iota(M^n)$ is again a flat cylinder and ι is a homethety.*

THEOREM 6. *Let M^n be the complete, simply connected space of constant negative curvature $n \geq 4$. Then M^n cannot be conformally analytically immersed as a complete hypersurface in $\mathbf{R}^{n+1}$.*

Theorem 6 should be compared with the well-known result of Hilbert asserting nonexistence of a complete surface of constant negative curvature in $\mathbf{R}^3$. For the corresponding deeper results about conformal immersions of Riemann surfaces see the works of Osserman, Garsia, Klotz, and Rüedy, see Rüedy (5) and the references there.

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1. Cartan, E. (1917) "La déformation des hypersurfaces dans l'espace conforme réel a $n \geq 5$ dimensions," *Bull. Soc. Math. France*, **45**, 57–121, especially -23–28.
2. Chern, S. S. & Simons, J. (1971) "Some Cohomology Classes in Principal Fiber Bundles," *Proc. Nat. Acad. Sci. USA* **68**, 791–794.
3. Kuiper, N. H. (*a*) (1949) "On conformally-flat spaces in the large," *Ann. Math.* **50**, 916–924; (*b*) (1950) "On compact conformally Euclidean spaces of dimension >2," *Ann. Math.* **52**, 478–490.
4. Schouten, J. A. (1921) "Über die konforme Abbildung...," *Math. Zeit.* **11**, 58–88, especially §23.
5. Rüedy, R. A. (1971) "Embeddings of open Riemann surfaces," *Comm. Math. Helv.* **46**, 214–225.